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$\mathrm{F}_4(2)$ AND ITS AUTOMORPHISM GROUP

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ABSTRACT. We present an identification theorem for the groups $\mathrm{F}_4(2)$ and $\mathrm{Aut}(\mathrm{F}_4(2))$ based on the structure of the centralizer of an element of order 3.

1. Introduction

In the classification of the finite simple groups a fundamental role was played by Timmesfeld's work on groups which contain a large extraspecial 2-subgroup [23]. Timmesfeld determined the structure of the normalizer of such a subgroup and following this achievement several authors contributed to the classification of all the simple groups which contain a large extraspecial 2-subgroup.

The notion of a large extraspecial 2-subgroup of a group is generalized in the work of Meierfrankenfeld, Stellmacher and the second author [13] to the concept of a large p-subgroup where p is an arbitrary prime. The definition of a large p- subgroup is as follows: given a finite group G, a p-subgroup Q of G is *large* if and only if

(L1) $Q = F^*(N_G(Q))$; and
(L2) for all non-trivial subgroups U of $Z(Q)$, $N_G(U) \leq N_G(Q)$.

Recall that condition (L1) is equivalent to $Q = O_p(N_G(Q))$ and $C_G(Q) \leq Q$. If Q is extraspecial and $p = 2$ this definition coincides with Timmesfeld's definition of a large extraspecial 2-group. The classification of groups with a large p-subgroup is sometimes called the MSS-project. The first step of this project is [13], where in contrast to the work of Timmesfeld, it is not the normalizer of Q which is determined but rather structural information about the maximal p-local subgroups of G which are not contained in $N_G(Q)$ is provided.

Suppose now that Q is a large subgroup of a group G and let S be a Sylow p-subgroup of G containing Q. It is an elementary exercise to show that $F^*(N_G(U)) = O_p(N_G(U))$ for all non-trivial normal subgroups U of S ([18, Lemma 2.1]). Groups which satisfy this property are said to be of *parabolic characteristic* p. If $F^*(N_G(U)) = O_p(N_G(U))$

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for all $1 \neq U \leq S$, then G is of *local characteristic* p (also called characteristic p-type). In [13] it is assumed that G has local characteristic p. However, there is work in progress which aims to remove this assumption, and so all the successor articles to [13] will be produced under the weaker hypothesis that the group under investigation has a large p-subgroup. One reason for this is that, as mentioned above, a group with a large p-subgroup is of parabolic characteristic p, while demonstrating that a group has local characteristic p may well be hard to verify in applications.

Nevertheless [13] provides us with some p-local structure of the group G and this is all that we require for the next step of the programme in which we aim to recognize G up to isomorphism. For this recognition we typically build a geometry upon which a subgroup of G acts. This means that we take some of the p-local subgroups of G which contain S and consider the subgroup H of G generated by them. The p-local subgroups are selected so that $O_p(H) = 1$. As the generic simple groups with a large p-subgroup are Lie type groups in characteristic p, in many cases we will be able to show that the coset geometry determined by the p-local subgroups in H is a building. The recognition of H is then achieved with help of the classification of buildings of spherical type [24, 25]. At this stage, as a third step of the programme, we would like to show that $G = H$. There is a general approach to achieve this goal. Since H contains S, it also contains Q and so we are able to identify Q as a subgroup of H. Typically $Q = F^*(N_H(R))$ for some root group R in H. We can then determine the structure of $N_G(Q)$. The aim is to show that $N_G(Q) = N_H(Q)$ and from this further show that $N_G(U) = N_H(U)$ for all $1 \neq U \trianglelefteq S$. The final step is to show that, if H is a proper subgroup of G, then H is strongly p-embedded in G and this contradicts the main results in [3] and [21].

However there are situations where it cannot be shown that $N_G(Q) = N_H(Q)$. This happens most frequently when $p = 2$ or 3 and $N_H(Q)$ is soluble. For the final stage of the project one has to analyze exactly these more troublesome configurations; that is determine all the groups G where $F^*(H)$ is a group of Lie type in characteristic p containing a Sylow p-subgroup S of G, $N_H(Q)$ is soluble and $N_H(Q) \neq N_G(Q)$. There are several configurations where this phenomenon arises. For example when $p = 3$ we have $H \cong \mathrm{P}\Omega_6^-(3)$ contained in $G \cong \mathrm{U}_6(2)$. Similarly, there are containments $\mathrm{P}\Omega_6^+(3)$ in $\mathrm{F}_4(2)$, $\mathrm{P}\Omega_7(3)$ in ${}^2\mathrm{E}_6(2)$ and $\mathrm{M}(22)$, and $\mathrm{P}\Omega_8^+(3)$ in $\mathrm{M}(23)$ and F_2. In all these cases Q is an extraspecial 3-group and $N_H(Q)$ is soluble. In a series of papers [17, 19, 20], the larger groups in this list are determined from the approximate structure of the centralizer of an element of order 3, or equivalently from the structure of $N_G(Q)$. In this paper we identify $\mathrm{F}_4(2)$ from the approximate structure of the centralizer of a 3-element. We are motivated by the embedding of $\mathrm{P}\Omega_6^+(3)$ in $\mathrm{F}_4(2)$, but we do not assume that G contains this group as we hope that our work can find broader application. We therefore just assume certain important structural information about the normalizer of Q and, as a consequence, this present article is independent of the results in [13].

This contribution should also be viewed as a companion to the authors' earlier work [17] in which the groups G with $\mathrm{PSU}_6(2) \le G \le \mathrm{Aut}(\mathrm{PSU}_6(2))$ are characterised by such information and this is a second reason why we make no additional assumption on the embedding of $\mathrm{P}\Omega_6^+(3)$ in the present article. Indeed, in such groups, the centralizer of a 3-element has a similar structure to that in $\mathrm{F}_4(2)$ or $\mathrm{Aut}(\mathrm{F}_4(2))$ but in these groups $Z(Q)$ is weakly closed in Q, while in $\mathrm{F}_4(2)$ and its automorphism group it is not. (Recall, for subgroups $X \le Y \le L$, we say X is *weakly closed* in Y with respect to L provided that if $g \in L$ and $X^g \le Y$, then $X^g = X$.) Unfortunately the arguments in these two situations are quite different. The theorems proved in [17] and in this article are employed in [18] to identify the corresponding groups.

We now make precise what we mean by the approximate structure of the centralizer of an element of order 3 in $\mathrm{PSU}_6(2)$ or $\mathrm{F}_4(2)$.

Definition 1.1. *We say that X is similar to a 3-centralizer in a group of type $\mathrm{PSU}_6(2)$ or $\mathrm{F}_4(2)$ provided the following conditions hold.*

(i) *$Q = F^*(X)$ is extraspecial of order 3^5 and $Z(F^*(X)) = Z(X)$; and*

(ii) *X/Q contains a normal subgroup isomorphic to $\mathrm{Q}_8 \times \mathrm{Q}_8$.*

Our main theorem is as follows.

Theorem 1.2. *Suppose that G is a group, $Z \le G$ has order 3. If $C_G(Z)$ is similar to a 3-centralizer in a group of type $\mathrm{PSU}_6(2)$ or $\mathrm{F}_4(2)$ and Z is not weakly closed in $F^*(C_G(Z))$, then $G \cong \mathrm{F}_4(2)$ or $\mathrm{Aut}(\mathrm{F}_4(2))$.*

Combining Theorem 1.2 and the main theorem from [17] we obtain the following statement.

Theorem 1.3. *Suppose that G is a group, $Z \le G$ has order 3. If $C_G(Z)$ is similar to a 3-centralizer in a group of type $\mathrm{PSU}_6(2)$ or $\mathrm{F}_4(2)$ and Z is not weakly closed in a Sylow 3-subgroup of $C_G(Z)$ with respect to G, then either $F^*(G) \cong \mathrm{F}_4(2)$ or $F^*(G) \cong \mathrm{PSU}_6(2)$.*

For groups G with $C_G(Z)$ of type $\mathrm{PSU}_6(2)$ or $\mathrm{F}_4(2)$, the different G-fusion of Z in $C_G(Z)$ manifests itself in the subgroup structure of G very quickly. Indeed, if we let S be a Sylow 3-subgroup of $C_G(Z)$

and $Q = F^*(C_G(Z))$, then we easily determine that $S \in \mathrm{Syl}_3(G)$ and the Thompson subgroup J of S has order 3^4 or 3^5 when Z is weakly closed in Q, whereas, it has order 3^4 if Z is not weakly closed in Q. More strikingly, setting $L = N_G(J)$, we have $F^*(L/Q) \cong \Omega_4^-(3)$ in the first case and in the second case $L/Q \cong \Omega_4^+(3)$.

The paper is set out as follows. In Section 2 we gather pertinent information about that natural and spin modules for $\mathrm{Sp}_6(2)$ and the natural and orthogonal $\mathrm{SU}_4(2)$-module as well as collect together further identification theorems and results which we shall require for the proof of Theorem 1.2. In Section 3 we present Theorem 3.3 which will be used to identify a subgroup P of our target group which is isomorphic to $\mathrm{F}_4(2)$. The proof of Theorem 3.3 involves the construction of a building of type $\mathrm{F}_4(2)$ on which P acts faithfully. The proof of the main theorem commences in Section 4. Thus we assume that G satisfies the hypothesis of Theorem 1.2 and set $M = N_G(Z)$. We remark here that the information that is developed as the proof of Theorem 1.2 unfolds becomes information about the groups $\mathrm{F}_4(2)$ and $\mathrm{Aut}(\mathrm{F}_4(2))$ once the theorem is proved. The initial objective of Section 4 is to determine more information about the structure of M. This is achieved by exploiting the fact that Z is not weakly closed in $Q = O_3(M)$. The first significant result is presented in Lemma 4.8 where it is shown that

$$M/Q \approx (\mathrm{Q}_8 \times \mathrm{Q}_8).\mathrm{Sym}(3) \text{ or } (\mathrm{Q}_8 \times \mathrm{Q}_8).(2 \times \mathrm{Sym}(3)).$$

In Section 4, we then move on, in Lemma 5.3, to the determination of L as described in the previous paragraph. At this stage we have shown that $L \approx 3^4 : \mathrm{GO}_4^+(3)$ or $3^4 : \mathrm{CO}_4^+(3)$. Thus J supports a quadratic form and G-fusion of elements in J is controlled by L. This allows us to parameterize the non-trivial cyclic subgroups of J as singular, plus and minus (the latter two types are fused when $L \approx 3^4 : \mathrm{CO}_4^+(3)$) and also the five types of subgroups of order 9 which we label Type S, Type D+, Type D-, Type N+ and Type N- (the notation is chosen to indicate that the groups are singular, degenerate with three plus groups, degenerate with three minus groups, non-degenerate of plus-type and non-degenerate of minus-type).

We let ρ_1 and ρ_2 be elements of $Q \cap J$ each centralized by a Q_8 (the quaternion group of order 8) subgroup of M and one generating a plus type and the other a minus type cyclic subgroup of J. In Section 6, we show that $C_G(\rho_1) \cong C_G(\rho_2) \cong 3 \times \mathrm{SU}_4(2)$ or $3 \times \mathrm{Sp}_6(2)$. (See Lemmas 6.3 and 6.4.) It is the latter possibility that actually arises in our target groups. There is related work in [6] that we might refer to at this stage but they assume that G is of characteristic 2-type.

We let r_1 and r_2 be central involutions in the subgroup of $C_G(Z)$ isomorphic to $\mathrm{Q}_8 \times \mathrm{Q}_8$ which do not invert Q/Z and, for $i = 1, 2$, we set $K_i = C_G(r_i)$. Again when $L \approx \mathrm{CO}_4^+(3)$ these groups are conjugate. At this stage we know that r_i centralizes the (simple) component of $C_G(\rho_i)$. The heart of the proof of Theorem 1.2 is contained in Sections 7, 8, 9 and 10 where we determine the structure of K_i. Thus the aim is to show that K_1 and K_2 have shape $2^{1+6+8}.\mathrm{Sp}_6(2)$ where $O_2(K_1)$ and $O_2(K_2)$ are commuting products of an extraspecial group of order 2^9 and an elementary abelian group of order 2^7.

We begin our construction of K_i by determining a large 2-group Σ_i which is normalized by $I_i = C_J(r_i)$. It turns out that Σ_i is the extraspecial 2-group of order 2^9 and plus type we are seeking. In the case that $C_G(\rho_i) \cong 3 \times \mathrm{SU}_4(2)$, we are able to show that in fact $K_i = N_G(\Sigma_i)$ and $N_G(\Sigma_i)/\Sigma_i \cong \mathrm{Aut}(\mathrm{SU}_4(2))$ or $\mathrm{Sp}_6(2)$ and this leads to a contradiction as explained in Lemma 8.2. Thus we enter Section 9 knowing that $C_G(\rho_1) \cong C_G(\rho_2) \cong 3\times\mathrm{Sp}_6(2)$. On the other hand Σ_i is far from being a maximal signalizer for I_i. Thus is Section 9 we construct an even larger signalizer which in the end is a product $\Gamma_i = \Sigma_i\Upsilon_i$ where Υ_i is an elementary abelian group of order 2^7. Thus Γ_i has order 2^{15} and in fact $\Upsilon_i = Z(\Gamma_i)$ and this is proved in Lemma 9.3. We show that $N_G(\Gamma_i)/\Gamma_i \cong \mathrm{Sp}_6(2)$ in Lemma 9.6. The final hurdle requires that we show that $K_i = N_G(\Gamma_i)$. This is proved in Lemma 10.8 and requires a sequence of lemmas which begins by showing that Υ_i is strongly closed in Γ_i with respect to K_i and culminates in the statement that Υ_i is strongly closed in a Sylow 2-subgroup of K_i with respect to K_i. At this stage we apply Lemma 2.19 which is essentially Goldschmidt's Strongly Closed Abelian 2-subgroup Theorem [5] to conclude that $K_i = N_G(K_i) \approx 2^{1+6+8}.\mathrm{Sp}_6(2)$. Our final section exploits Theorem 3.3 to produce a subgroup P of G with $P \cong \mathrm{F}_4(2)$. We show that a group closely related to P is strongly 3-embedded in G and finally apply Holt's Theorem [10] in the form presented in Lemma 2.20 to conclude the proof of the Theorem 1.2.

Throughout this article we follow the now standard Atlas [4] notation for group extensions. Thus $X{}^{\cdot}Y$ denotes a non-split extension of X by Y, $X{:}Y$ is a split extension of X by Y and we reserve the notation $X.Y$ to denote an extension of undesignated type (so it is either unknown, or we don't care). Our notation follows that in [1], [7] and [8]. We use the definition of signalizers as given in [8, Definition 23.1]. For odd primes p, the extraspecial groups of exponent p and order p^{2n+1} are denoted by p_+^{1+2n}. The extraspecial 2-groups of order 2^{2n+1} are denoted by 2_+^{1+2n} if the maximal elementary abelian subgroups have order 2^{1+n} and otherwise we write 2_-^{1+2n}. We expect our notation for specific groups is self-explanatory. For a subset X of a group G, X^G denotes the set of G-conjugates of X. If $x, y \in H \le G$, we write $x \sim_H y$ to indicate that x and y are conjugate in H. Often we shall give suggestive descriptions of groups which indicate the isomorphism type of certain composition factors. We refer to such descriptions as the *shape* of a group. Groups of the same shape have normal series with isomorphic sections. We use the symbol $\approx$ to indicate the shape of a group.

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2. Preliminaries

In this section we lay out certain facts about the groups $\mathrm{Sp}_6(2)$ and $\mathrm{Aut}(\mathrm{U}_4(2))$ which play a pivotal role in the proof of our main theorem. We also present other background results that are of key importance to our investigations.

Lemma 2.1. *Suppose that* $X \cong \mathrm{Sp}_6(2)$ *or* $\mathrm{Aut}(\mathrm{SU}_4(2))$. *Then there is a unique irreducible* $\mathrm{GF}(2)X$*-module of dimension* 6 *and a unique irreducible* $\mathrm{GF}(2)X$*-module of dimension 8. All the other non-trivial irreducible* $\mathrm{GF}(2)X$*-modules have dimension at least* 9.

Proof. This is well known. See [12]. $\square$

In this section U will denote the $\mathrm{Aut}(\mathrm{SU}_4(2))$ natural module and the $\mathrm{Sp}_6(2)$ spin module of dimension 8 and V will be the $\mathrm{Aut}(\mathrm{SU}_4(2))$ orthogonal module and the $\mathrm{Sp}_6(2)$ natural module of dimension 6.

For $X \cong \mathrm{Sp}_6(2)$, let X_1, X_2 and X_3 be the minimal parabolic subgroups of X containing a fixed Sylow 2-subgroup S. Set $X_{ij} = \langle X_i, X_j \rangle$ where $1 \le i < j \le 3$ and fix notation so that

$$X_{12}/O_2(X_{12}) \cong \mathrm{SL}_3(2),$$
$$X_{23}/O_2(X_{23}) \cong \mathrm{Sp}_4(2) \text{ and}$$
$$X_{13}/O_2(X_{13}) \cong \mathrm{SL}_2(2) \times \mathrm{SL}_2(2).$$

There are three conjugacy classes of elements of order 3 in X. Let τ_1, τ_2 and τ_3 be representatives of these classes and choose so that on the natural $\mathrm{Sp}_6(2)$-module V, for $1 \le i \le 3$, $\dim[V, \tau_i] = 2i$.

		Centralizer in $\mathrm{Aut}(\mathrm{SU}_4(2))$	Centralizer in $\mathrm{Sp}_6(2)$	$\dim C_U(u_j)$	$\dim C_V(u_j)$
a_2	u_1	$2_+^{1+4}.(\mathrm{SL}_2(2)\times\mathrm{SL}_2(2))$	$2^{1+2+4}.(\mathrm{SL}_2(2)\times\mathrm{SL}_2(2))$	6	4
b_3	u_2	$2\times(\mathrm{Sym}(4)\times 2)$	$2^7.3$	4	3
b_1	u_3	$2\times\mathrm{Sp}_4(2)$	$2^5.\mathrm{Sp}_4(2)$	4	5
c_2	u_4	$2^6.3$	$2^8.\mathrm{SL}_2(2)$	4	4

TABLE 1. Involutions in $\mathrm{Sp}_6(2)$ and $\mathrm{Aut}(\mathrm{SU}_4(2))$. The involutions in the first row are the *unitary transvections.* The involutions labeled with "b" those which are in $\mathrm{Aut}(\mathrm{SU}_4(2))\setminus\mathrm{SU}_4(2)$.

Lemma 2.2. *Suppose that $Y\cong\mathrm{Aut}(\mathrm{SU}_4(2))$ and that $X\cong\mathrm{Sp}_6(2)$ with $Y\le X$. Assume that V and U are the faithful $\mathrm{GF}(2)X$-modules of dimension 6 and 8 respectively.*

(i) *X and Y each have four conjugacy classes of involutions and for each involution $u\in X$ we have $u^X\cap Y$ is a conjugacy class in Y. In column one of Table 1 we provide the Suzuki names (see* [2, page 16]*) for each class of involutions.*
(ii) *The shape of the centralizers of involutions in X and Y is given in Table 1.*
(iii) *For each involution in $u\in X$, $\dim C_V(u)$ and $\dim C_U(u)$ is given in Table 1.*
(iv) *X does not contain any subgroup of order 2^4 in which all the involutions are conjugate.*
(v) *X does not contain an extraspecial subgroup of order 2^7.*
(vi) *If x is an involution of type b_1, then a Sylow 3-subgroup of $C_Y(u)$ contains two conjugates of $\langle\tau_1\rangle$ and two conjugates of $\langle\tau_2\rangle$.*
(vii) *$E=\langle\tau_1,\tau_2,\tau_3\rangle$ is the Thompson subgroup of a Sylow 3-subgroup of G and every element of order 3 is X-conjugate (Y-conjugate) to an element of E.*

Proof. Parts (i)-(iii) follow from [17, Proposition 2.12, and Table 1].

Suppose that $A\le X$ has order 2^4 and that all the non-trivial elements are conjugate in X. We use the character table of X given in [4, page 47]. Let χ be an irreducible character of X. Then, as $(\chi|_A,1_A)\ge 0$, we have

$$(\chi|_A,1_A)=\frac{1}{|A|}\sum_{a\in A}\chi(a)\ge 0.$$

Taking χ to be the degree 7 character we see that all the non-trivial elements in A are in Suzuki class c_2 (Atlas [4] $2C$). Now considering the character of degree 35 denoted χ_7 in [4] we obtain a contradiction.

Let E be extraspecial of order 2^7. Since X has a faithful 7-dimensional representation in characteristic 0 and the smallest such representation of E is 8-dimensional, E is not isomorphic to a subgroup of X.

Part (vi) follows from the action of $\mathrm{Sp}_4(2)$ on the natural module for $\mathrm{Sp}_6(2)$ as $\mathrm{Sp}_4(2)$ contains no conjugates of τ_3.

Part (vii) is also elementary to verify.

□

Lemma 2.3. *Let $X \cong \mathrm{Sp}_6(2)$, S a Sylow 2-subgroup of X and V be the $\mathrm{Sp}_6(2)$ natural module. Then the following hold.*

(i) *X acts transitively on the non-zero vectors in V.*

(ii) *V is uniserial as an S-module.*

(iii) *Suppose that, for $1 \le i \le 3$, V_i is an S-invariant subspace of V of dimension i. Then $X_{23} = N_X(V_1)$ and X_{23} acts naturally as $\mathrm{Sp}_4(2)$ on $V_1^\perp/V_1$, $X_{13} = N_X(V_2)$, $O^2(X_3)$ centralizes V_2 and $V/V_2^\perp$, and $O^2(X_1)$ centralizes $V_2^\perp/V_2$ and $X_{12} = N_X(V_3)$ and acts naturally on both V_3 and V/V_3.*

Proof. These are all well known facts about the action of X on V. See for example [15, Lemma 14.37] for (i) and (ii). □

Lemma 2.4. *Let $X \cong \mathrm{Sp}_6(2)$, S a Sylow 2-subgroup of X and U be the $\mathrm{Sp}_6(2)$ spin module.*

(i) *X has exactly two orbits on the non-zero vectors of U one of length 135 and one of length 120.*

(ii) *$N_X(C_U(S)) = X_{12}$ and $C_U(S) = C_U(O_2(X_{12}))$.*

(iii) *If $U_2 \le U$ is S-invariant of dimension 2, then $N_X(U_2) = X_{13}$ and $O^2(X_1)$ centralizes U_2.*

Proof. See [17, Proposition 2.12]. □

Lemma 2.5. *Suppose that $X \cong \mathrm{Sp}_6(2)$ and V is the natural module for X. Let $P = X_{13}$, $T \in \mathrm{Syl}_3(P)$ and $Q = O_2(P)$.*

(i) *$P/Q \cong \mathrm{SL}_2(2) \times \mathrm{SL}_2(2)$.*

(ii) *The subgroups of order 3 in T are as follows: there are two subgroups Z_1 and Z_2 which are X-conjugate to $\langle \tau_3 \rangle$, one subgroup which is X-conjugate to $\langle \tau_1 \rangle$ (which we suppose is $\langle \tau_1 \rangle$) and one subgroup which is X-conjugate to $\langle \tau_2 \rangle$. The two subgroups of T which are conjugate to $\langle \tau_3 \rangle$ are conjugate in $N_P(T)$.*

(iii) *$C_Q(Z_1) \cong C_Q(Z_2) \cong \mathrm{Q}_8$ and $[C_Q(Z_1), C_Q(Z_2)] = 1$.*

(iv) *$C_T(Z(Q)) = \langle \tau_1 \rangle$ and $C_Q(\tau_1) = Z(Q)$.*

(v) *If* $U \leq Q$ *has order* 2^3 *and if* U *is* T*-invariant, then either* $U = C_Q(Z_1)$, $U = C_Q(Z_2)$ *or* $U = Z(Q)$.
(vi) *Let* $Q' = \langle t \rangle$. *Then* $t^X \cap Q \not\subseteq Z(Q)$.

Proof. Let Y be the P-invariant isotropic 2-space in V. Then P preserves $0 < Y < Y^\perp < V$. Let I be a hyperbolic line and $J = I^\perp$ be chosen so $Y \leq J$. Then the decomposition $I \perp J$ is preserved by $\mathrm{Sp}_2(2) \times \mathrm{Sp}_4(2)$ and the subgroup K of this group which leaves Y invariant has shape $\mathrm{Sp}_2(2) \times (2 \times 2^2).\mathrm{SL}_2(2) \cong \mathrm{SL}_2(2) \times 2 \times \mathrm{Sym}(4)$. In particular, we now have (i) holds. Furthermore, we may suppose the first factor of K contains $\langle \tau_1 \rangle$ while the second factor contains $\langle \tau_2^* \rangle$, an X-conjugate of $\langle \tau_2 \rangle$, acting fixed point freely on J. Set $T = \langle \tau_1, \tau_2^* \rangle$. Since τ_1 is inverted in the first factor of K, we see the two diagonal products $\tau_1\tau_2^*$ and $\tau_1^2\tau_2^*$ are conjugate in $N_P(T)$. Furthermore these elements act fixed point freely on V and so are X-conjugate to τ_3. This is (ii).

Now consider Q. We know this group has order 2^7. We further have $Q \cap K = O_2(K)$ centralizes $Y + I = Y^\perp$. Consequently $Q \cap K$ is normal in P and as $[V, Q, Q \cap K] = [V, Q \cap K, Q]$ we additionally have $K \cap Q \leq Z(Q)$. Note that $\langle \tau_1 \rangle$ centralizes $Q \cap K$. Now $C_P(\tau_2^*)$ is contained in K and so we see $C_Q(\tau_2^*) = Z(K)$ has order 2. Now the centralizer in X of τ_3 supports a $\mathrm{GF}(4)$ structure and is isomorphic to $\mathrm{SU}_3(2)$. It follows that $\tau_1\tau_2^*$ and $\tau_1^2\tau_2^*$ can centralize only quaternion subgroups of order 8 in Q. Since $C_Q(\tau_1\tau_2^*)$ and $C_Q(\tau_1^2\tau_2^*)$ both centralize $Z(K)$ and $|Q| = 2^7$ we have $C_Q(\tau_1\tau_2^*) \cong C_Q(\tau_1^2\tau_2^*) \cong \mathrm{Q}_8$ and $C_Q(\tau_1\tau_2^*)' = Z(K)$. Putting $Q_1 = C_Q(\tau_1\tau_2^*)C_Q(\tau_1^2\tau_2^*)$ we have Q_1 is T-invariant. Now $Q = C_Q(\tau_1\tau_2^*)C_Q(\tau_1^2\tau_2^*)(Q \cap K)$,

$$[Q, \tau_1] = [C_Q(\tau_1\tau_2^*), \tau_1][C_Q(\tau_1^2\tau_2^*), \tau_1] = Q_1$$

is a normal subgroup of Q and $Q_1 \cap (Q \cap K) \leq Z(K)$. Thus Q_1 is extraspecial and $Q' = Z(K)$ which has order 2. In addition, $Q = C_Q(\tau_1\tau_2^*)[Q, \tau_1\tau_2^*]$ with $C_Q(\tau_1\tau_2^*) \cap [Q, \tau_1\tau_2^*] = Z(K)$. Since

$$[C_Q(\tau_1\tau_2^*), Q, \tau_1\tau_2^*] \leq [Z(K), \tau_1\tau_2^*] = 1$$

and $[C_Q(\tau_1\tau_2^*), \tau_1\tau_2^*, Q] = 1$, we also have $[C_Q(\tau_1\tau_2^*), [Q, \tau_1\tau_2^*]] = 1$ by the Three Subgroup Lemma. In particular, as $[Q, \tau_1\tau_2^*] = C_Q(\tau_1^2\tau_2^*)(Q \cap K)$, we now have (iii) and (iv) hold. If U is of order 2^3 and is T-invariant, then $C_T(U) > 1$ and so (v) also follows from the above discussion. To prove (vi), we start with a transvection $r \in Z(Q)$. By Table 1 we have $E = O_2(C_X(r))$ is elementary abelian of order 2^5. Now $|E \cap Q| \geq 2^3$. If $E \cap Q \leq Z(Q)$, then, as $E \leq C_{N_X(Q)}(E \cap Q)$, we get $|E \cap Q| \geq 2^4$, a contradiction. Hence $E \cap Q \not\leq Z(Q)$. Now as $N_X(E)$ acts transitively on $E/\langle r\rangle$, we have any coset of $\langle r\rangle$ in E contains a conjugate of t. In particular $t^X \cap E \cap Q \not\subseteq Z(Q)$. $\square$

Lemma 2.6. *Let* $Y = \mathrm{Aut}(\mathrm{SU}_4(2))$ *and* V *be the natural* $\mathrm{O}_6^-(2)$*-module. Then there is no elementary abelian subgroup* E *of order* 8 *in* Y *such that* $|V : C_V(E)| \le 4$.

Proof. Suppose false and let E be such a subgroup of order 8. From Table 1 we see E cannot contain elements of type b_3. If $E \not\le Y'$, then E contains exactly four elements of type b_1. As there are at most three hyperplanes in V containing $C_V(E)$, two of these elements have to centralize the same hyperplane of V. But then their product, which is an involution in $E \cap Y$, also centralizes this hyperplane. As $\Omega_6^-(2)$ does not contain transvections, we have $E \le Y'$. Therefore $|V : C_V(E)| = 4$ and $C_V(E) = C_V(e)$ for all $e \in E^\#$. As $C_V(e) = [V, e]^\perp$ we also have $[V, e] = [V, E]$ for all $e \in E^\#$ which means all the involutions in E are conjugate. Now we use the character table of $\mathrm{SU}_4(2)$ as in the proof of Lemma 2.2(iv) to obtain a contradiction. $\square$

Recall that a faithful $\mathrm{GF}(p)G$-module is an *F-module* provided there exists a non-trivial elementary abelian p-subgroup $A \le G$ such that $|V : C_V(A)| \le |A|$. The subgroups $A \le G$ with $|V : C_V(A)| \le |A|$ are called *offenders*.

Lemma 2.7. *Suppose that* $X \cong \mathrm{Sp}_6(2)$ *or* $\mathrm{Aut}(\mathrm{SU}_4(2))$ *and* W *is a* $\mathrm{GF}(2)X$*-module of dimension* 14 *which has exactly two composition factors one of dimension 6 and one of dimension* 8*. Then* W *is not an* F*-module.*

Proof. Suppose that $A \le X$ is an offender on W. Then $|A| \ge |W : C_W(A)|$. From Table 1, for $a \in A$, we read $|A| \ge |W : C_W(a)| \ge 2^4$. Since the 2-rank of X is at most 6, we also have that A does not contain any involutions of type b_3.

Suppose that $|A| = 2^4$. Then all the involutions in A must be of type a_2. This contradicts Lemma 2.2(iv). Hence $|A| \ge 2^5$ and $X \cong \mathrm{Sp}_6(2)$ as the 2-rank of $\mathrm{Aut}(\mathrm{SU}_4(2))$ is 4 (see [17, Proposition 2.12 (x)]). We use the notation for involutions from Table 1. We may as well suppose $A \le C_X(u_3)$. Then as the 2-rank of $\mathrm{Sp}_4(2)$ is 3, we have $A \cap O_2(C_X(u_3)) \ne 1$. Since $|C_U(O_2(C_X(u_3)))| = 2^4$ and $|C_V(O_2(C_X(u_3)))| = 2$ certainly $A \ne O_2(C_X(u_3))$. Now $O_2(C_X(u_3))$ contains 15 elements from u_1^X, 15 elements from u_4^X and one element from u_3^X and multiplication by u_3 maps $u_1^X \cap O_2(C_G(u_3))$ to $u_4^X \cap O_2(C_X(u_3))$. Thus, if A contains a conjugate of u_3, then $A \cap u_i^X \ne \emptyset$ for $i = 1, 3, 4$. As $|A| = 2^5$, A does not consist purely of elements of elements from class u_1^X by Lemma 2.2

(iv) and consequently we must have elements from u_4^X in X. It follows now from Table 1 that $|A| = 2^6$. There is a unique such elementary abelian subgroup in a Sylow 2-subgroup of X and its normalizer is a plane stabiliser in the action of X on V. But then $|W : C_W(A)| \geq 2^{10}$ which is a contradiction. $\square$

Lemma 2.8. *Suppose that $X \cong \mathrm{Sp}_6(2)$, W is a 7-dimensional $\mathrm{GF}(2)X$-module with $W/C_W(X)$ the natural $\mathrm{Sp}_6(2)$-module. If $S \in \mathrm{Syl}_2(X)$, then $C_W(S) > C_W(X)$.*

Proof. Consider the subgroup $K = K_1 \times K_2$ of X which preserves the decomposition of $W/C_W(X)$ in to a perpendicular sum of a non-degenerate 2-space $A/C_W(X)$ and a non-degenerate 4-space $B/C_W(X)$ with $K_1 \cong \mathrm{Sp}_2(2)$ and $K_2 \cong \mathrm{Sp}_4(2)$. Let t be an involution in K_1. Since $\dim A = 3$, we have $\dim[A,t] = 1$. Furthermore $B/C_B(t) \cong [B,t]$ as K_2-modules and so we must have $[B,t] = 0$. Thus $[W,t] = [A,t] + [B,t] = [A,t]$ has dimension 1 and so t is a transvection on W. Let $P = C_X(t)$. Then P contains a Sylow 2-subgroup S of X. Since P centralizes $[W,t]$ and $C_W(X)$, P centralizes $L = [W,t] + C_W(X)$ and so $L \leq C_W(S)$. $\square$

Theorem 2.9 (Prince)**.** *Suppose that Y is isomorphic to the centralizer of a 3-central element of order 3 in $\mathrm{PSp}_4(3)$ and that X is a finite group with a non-trivial element d such that $C_X(d) \cong Y$. Let $P \in \mathrm{Syl}_3(C_X(d))$ and E be the elementary abelian subgroup of P of order 27. If E does not normalize any non-trivial $3'$-subgroup of X and d is X-conjugate to its inverse, then either*

(i) $|X : C_X(d)| = 2$;
(ii) *X is isomorphic to* $\mathrm{Aut}(\mathrm{SU}_4(2))$; *or*
(iii) *X is isomorphic to* $\mathrm{Sp}_6(2)$.

Proof. See [22, Theorem 2]. $\square$

Lemma 2.10. *Suppose that X is a group of shape $3_+^{1+2}.\mathrm{SL}_2(3)$, $O_2(X) = 1$ and a Sylow 3-subgroup of X contains an elementary abelian subgroup of order 3^3. Then X is isomorphic to the centralizer of a non-trivial 3-central element in $\mathrm{PSp}_4(3)$.*

Proof. See [14, Lemma 6]. $\square$

Lemma 2.11. *Suppose that F is a field, V is an n-dimensional vector space over F and $G = \mathrm{GL}(V)$. Assume that q is quadratic form of Witt index at least 1 and with non-degenerate associated bilinear form f, where, for $v, w \in V$, $f(v,w) = q(v+w) - q(v) - q(w)$. Let $\mathcal{S}$ be the set of singular 1-dimensional subspaces of V with respect to q. Then the stabiliser in G of $\mathcal{S}$ preserves q up to similarity.*

Proof. See [16, Lemma 2.10]. □

Lemma 2.12. *Suppose that p is an odd prime, $X = \mathrm{GL}_4(p)$ and V is the natural $\mathrm{GF}(p)G$-module. Let $A = \langle a, b\rangle \le X$ be elementary abelian of order p^2 and assume that $[V, a] = C_V(b)$ and $[V, b] = C_V(a)$ are distinct and of dimension 2. Let $v \in V \setminus [V, A]$. Then A leaves invariant a non-degenerate quadratic form with respect to which v is a singular vector. In particular, X contains exactly two conjugacy classes of subgroups such as A. One is conjugate to a Sylow p-subgroup of $\mathrm{GO}_4^+(p)$ and the other to a Sylow p-subgroup of $\mathrm{GO}_4^-(p)$.*

Proof. See [16, Lemma 2.11]. □

The 4-dimensional orthogonal module of +-type will play a prominent role in the proof of our main theorem. We next introduce some notation which will be used in the proof.

Notation 2.13. *Let V be a 4-dimensional non-degenerate orthogonal space of +-type over $\mathrm{GF}(3)$. Assume that X is a non-zero subspace of V. Then $\mathcal{S}(X)$ is the set of singular 1-dimensional subspaces in X, $\mathcal{P}(X)$ the set of 1-dimensional subspaces of +-type in X and $\mathcal{M}(X)$ the set of 1-dimensional subspaces of −-type in X.*

Lemma 2.14. *Let X be a 3-dimensional subspace in a non-degenerate 4-dimensional orthogonal space of +-type over $\mathrm{GF}(3)$. Then $\mathcal{S}(X) \ne \emptyset$.*

Proof. See [1, 21.3]. □

We now introduce some additional notation:

Notation 2.15. *Let V be a 4-dimensional non-degenerate orthogonal space of +-type over $\mathrm{GF}(3)$ and E be a 2-dimensional subspace of V. The type of E is determined by the number of 1-dimensional subspaces of a given type in E. Thus we have*

Type S: $|\mathcal{S}(E)| = 4$.
Type D+: $|\mathcal{S}(E)| = 1$ *and* $|\mathcal{P}(E)| = 3$.
Type D-: $|\mathcal{S}(E)| = 1$ *and* $|\mathcal{M}(E)| = 3$.
Type N+: $|\mathcal{S}(E)| = 2$ *and* $|\mathcal{M}(E)| = |\mathcal{P}(E)| = 1$.
Type N-: $|\mathcal{P}(E)| = |\mathcal{M}(E)| = 2$.

Lemma 2.16. *Let V be a 4-dimensional non-degenerate orthogonal space over $\mathrm{GF}(3)$ of +-type and E be a 2-dimensional subspace of V. Then E is of one of the types in Notation 2.15.*

Proof. The subspaces of V of dimension 2 are either totally singular (S), degenerate with three elements of $\mathcal{P}(V)$ (D+), degenerate with three elements from $\mathcal{M}(V)$ (D-) , non-degenerate of plus type (N+), or non-degenerate of minus type (N-). □

Theorem 2.17. *Suppose that G is a finite group, Q is a subgroup of G and $H = N_G(Q)$. Assume that the following hold*

(i) $H/Q \cong \mathrm{Aut}(\mathrm{SU}_4(2))$ *or* $\mathrm{Sp}_6(2)$*;*
(ii) $Q = C_G(Q)$ *is a minimal normal subgroup of H and is elementary abelian of order* 2^8*;*
(iii) H *controls G-fusion of elements of H of order* 3*; and*
(iv) *if $g \in G \setminus H$ and $d \in H \cap H^g$ has order* 3*, then* $C_Q(d) = 1$*.*

Then $G = HO_{2'}(G)$*.*

Proof. This is [16, Theorem 3.1]. □

Lemma 2.18. *Suppose that G is a group, E is an extraspecial* 2*-group which is normal in G and $x \in G \setminus C_G(E)$ is an involution. If x is not E-conjugate to xe where $e \in Z(E)^\#$, then $C_E(x) \geq [E, x]$ and $[E, x]$ is elementary abelian.*

Proof. Certainly $C_{E/Z(E)}(x) \geq [E/Z(E), x]$. Therefore, if $C_E(x) \not\geq [E, x]$, then $[f, x, x] = e$ for some $f \in E$. Setting $w = [f, x]$ we then have $x^w = xe$ which contradicts our hypothesis on x. Hence $C_E(x) \geq [E, x]$.

We now show that every element of $[E, x]$ has order 2. Let $f \in [E, x]$. Then fe has the same order as f. Thus we may suppose that $f = [h, x]$ for some $h \in E$. As $[E, x] \leq C_E(x)$, $x[h, x] = [h, x]x$ and so

$$\begin{aligned} f^2 &= [h,x][h,x] = h^{-1}xhx[h,x] = h^{-1}xh[h,x]x \\ &= h^{-1}xhh^{-1}xhxx = 1 \end{aligned}$$

as required. This proves the lemma. □

For a group X with subgroups $A \leq Y \leq X$, we say that A is *strongly closed in Y with respect to X* provided $A^x \cap Y \leq A$ for all $x \in X$.

Lemma 2.19. *Suppose that K is a group, $O_{2'}(K) = 1$, A is an abelian* 2*-subgroup of K and A is strongly closed in $N_K(A)$ with respect to K. Assume that $F^*(N_K(A)/C_K(A))$ is a non-abelian simple group. Then $K = N_K(A)$.*

Proof. Set $L = \langle A^K \rangle$. Since $O_{2'}(K) = 1$, we have $O_{2'}(L) = 1$. By Goldschmidt [5, Theorem A], $L = O_2(L)E(L)$ and $A = O_2(L)\Omega_1(T)$ where $T \in \mathrm{Syl}_2(L)$ contains A. If $E(L) = 1$, then A is normal in K and we are done. Thus $E(L) \neq 1$. Goldschmidt additionally states that $E(L)$ is a direct product of simple groups of type $\mathrm{PSL}_2(q)$, $q \equiv 3, 5 \pmod 8$, ${}^2\mathrm{G}_2(3^a)$, $\mathrm{SL}_2(2^a)$, $\mathrm{PSU}_3(2^a)$, ${}^2\mathrm{B}_2(2^a)$ for some natural number a, or the sporadic simple group J_1. It follows from the structure of these groups that $N_L(A)$ is a soluble group which is not a 2-group. On the other hand, $N_L(A) = L \cap N_K(A)$ is a normal subgroup of $N_K(A)$. Since $F^*(N_K(A)/C_K(A))$ is a non-abelian simple group and $N_L(A)$ is soluble we now have $N_L(A) \le C_K(A)$ and this contradicts the structure of $E(L)$. Thus A is normal in K as claimed. $\square$

We will also need the following statement of Holt's Theorem [10].

Lemma 2.20. *Suppose that K is a simple group, P is a proper subgroup of K and r is a 2-central element of K. If $r^K \cap P = r^P$ and $C_K(r) \le P$, then $K \cong \mathrm{PSL}_2(2^a)$ $(a \ge 2)$, $\mathrm{PSU}_3(2^a)$ $(a \ge 2)$, ${}^2\mathrm{B}_2(2^a)$ $(a \ge 3$ and odd$)$ or $\mathrm{Alt}(n)$ $(n \ge 5)$ where in the first three cases P is a Borel subgroup of K and in the last case $P \cong \mathrm{Alt}(n-1)$.*

Proof. Set $\Omega = K/P$ and assume that $P < K$. The conditions $C_K(r) \le P$ and $r^K \cap P = r^P$ together imply that r fixes a unique point of Ω. Let J be the set of involutions of K which fix exactly one point of Ω. Since r is a 2-central element of K, any 2-group which fixes at least 3 points when it acts on Ω commutes with an element of J. Hence Holt's criterion (*) from [10] is satisfied. In addition, the simplicity of K yields $K = \langle r^K \rangle = \langle J \rangle$. Thus [10, Theorem 1] implies that K is isomorphic to one of the following groups $\mathrm{PSL}_2(2^n)$, $\mathrm{PSU}_3(2^n)$, ${}^2\mathrm{B}_2(2^n)$ ($n \ge 3$ and odd) or $\mathrm{Alt}(\Omega)$ where in the first three classes of groups the stabiliser P is a Borel subgroup and in the latter case it is $\mathrm{Alt}(\Omega \setminus \{P\})$. $\square$

For the final steps in the identification of $\mathrm{F}_4(2)$ we need information about its involutions and their centralizers.

Lemma 2.21. *The group $X = \mathrm{F}_4(2)$ has four conjugacy classes of involutions x_1, x_2, x_3 and x_4 three of which are 2-central. Furthermore we may assume that notation is chosen so that*

(i) $C_X(x_1) \cong C_X(x_2) \approx 2^{1+6+8}.\mathrm{Sp}_6(2)$;
(ii) $C_X(x_3) \approx 2^{1+1+4+1+4+4+1+4}.\mathrm{Sp}_4(2)$; *and*
(iii) $C_X(x_4) \approx 2^{[9]}.(\mathrm{SL}_2(2) \times \mathrm{SL}_2(2))$.

Proof. These facts can be found in Guterman [9, Section 3] (see also [2, Page 45]) . $\square$

3. IDENTIFYING $\mathrm{F}_4(2)$

The final step in the proof of Theorem 1.2 demands that we can identify $\mathrm{F}_4(2)$ or $\mathrm{Aut}(\mathrm{F}_4(2))$ from the structure of the centralizer of a certain 2-central involution. In this section we give such an identification. The centralizers of interest are the centralizers of the involutions x_1, x_2 in $\mathrm{F}_4(2)$ as given in Lemma 2.21 (i). Of course, we do not want to specify the isomorphism type of such a centralizer, but only the approximate shape of the group.

Definition 3.1. *We say the group U is similar to a 2-centralizer in a group of type* $\mathrm{F}_4(2)$ *if U has the following properties.*

(i) $U/O_2(U) \cong \mathrm{Sp}_6(2)$;
(ii) $O_2(U)$ *is an product of* $Z(O_2(U))$ *by an extraspecial group of order* 2^9, $Z(O_2(U))$ *is elementary abelian of order* 2^7; *and*
(iii) $U/O_2(U)$ *induces the natural module on* $Z(O_2(U))/O_2(U)'$ *and the spin module on* $O_2(U)/Z(O_2(U))$.

Definition 3.2. *Suppose that G is a group and assume that the following hold:*

(i) *For $i = 1, 2$, there are involutions x_i in G such that* $U_i = C_G(x_i)$ *is similar to a 2-centralizer in a group of type* $\mathrm{F}_4(2)$.
(ii) *There is a Sylow 2-subgroup T of U_1 such that* $Z(T) = \langle x_1, x_2\rangle$.

Then we say that U_1, U_2, T is an F_4 set-up in G.

Our identification theorem in this section is as follows:

Theorem 3.3. *If U_1, U_2, T is an F_4 set-up in G, then* $\langle U_1, U_2\rangle \cong \mathrm{F}_4(2)$.

For the remainder of this section we assume that U_1, U_2 and T is an F_4 set-up in G. Notice that because of Definition 3.1 (ii), for $i = 1, 2$, $O_2(U_i)' = \langle x_i\rangle$ has order 2. The first lemma details the relationship of U_1 with U_2.

Lemma 3.4. *The following hold:*

(i) $U_1 \cap U_2$ *contains* T;
(ii) $(U_1 \cap U_2)/O_2(U_1 \cap U_2) \cong \mathrm{Sp}_4(2)$;
(iii) $O_2(U_1 \cap U_2) = O_2(U_1)O_2(U_2)$; *and*
(iv) $Z(T) = Z(O_2(U_1)) \cap Z(O_2(U_2))$.

Proof. From part (ii) of the definition of an F_4 set-up in G, we have $T \le U_1 \cap U_2$. This proves (i).

Since $Z(U_i)/\langle x_i\rangle$ is a natural $U_i/O_2(U_i)$-module and $|Z(T)| = 4$, Lemma 2.8 implies $Z(T) \le Z(U_1) \cap Z(U_2)$. Therefore, by Lemma 2.3 (iii),

$$\begin{aligned}(U_1 \cap U_2)/O_2(U_1 \cap U_2) &= C_{U_1}(Z(T))/O_2(C_{U_1}(Z(T))\\ &= C_{U_2}(Z(T))/O_2(C_{U_1}(Z(T)) \cong \mathrm{Sp}_4(2).\end{aligned}$$

Hence (ii) holds.

Since

$$(O_2(U_1) \cap O_2(U_2))' \le O_2(U_1)' \cap O_2(U_2)' = \langle x_1\rangle \cap \langle x_2\rangle = 1,$$

$O_2(U_1) \cap O_2(U_2)$ is abelian. Therefore, as $O_2(U_1)$ contains an extraspecial subgroup of order 2^9, we have

$$|O_2(U_1) : O_2(U_1) \cap O_2(U_2)| \ge 2^4.$$

Furthermore, as $O_2(U_1)O_2(U_2)/O_2(U_1)$ is normal in $(U_1 \cap U_2)/O_2(U_1)$, $O_2(U_1 \cap U_2) = O_2(U_1)O_2(U_2)$ follows from Lemma 2.3 (iii). This is (iii).

Finally, since $O_2(U_1 \cap U_2)$ centralizes $Z(O_2(U_1)) \cap Z(O_2(U_2))$, we deduce $Z(T) = Z(O_2(U_1)) \cap Z(O_2(U_2))$ and this proves (iv). $\square$

Our method to prove Theorem 3.3 is to use the F_4 set-up U_1, U_2, T in G to construct a chamber system of type $\mathrm{F}_4(2)$ using the subgroup $P = \langle U_1, U_2\rangle$ of G. To accomplish this we first define P_1, P_2, P_3 to be subgroups of U_1 containing T such that $P_j/O_2(U_1)$, $j = 1, 2, 3$, are the minimal parabolic subgroups of $U_1/O_2(U_1)$ containing $T/O_2(U_1)$. We additionally let P_4 be such that $U_2 \geq P_4 \geq T$, $P_4 \not\leq U_1$ and $P_4/O_2(U_2)$ is a minimal parabolic subgroup of $U_2/O_2(U_2)$. For $\emptyset \neq \sigma \subseteq \{1, 2, 3, 4\}$ we set $P_\sigma = \langle P_j \mid j \in \sigma\rangle$.

We may assume that notation has been chosen so that

$$\begin{aligned} P_{12}/O_2(P_{12}) &\cong \mathrm{SL}_3(2); \\ P_{13}/O_2(P_{13}) &\cong \mathrm{SL}_2(2) \times \mathrm{SL}_2(2); \text{ and} \\ P_{23}/O_2(P_{23}) &\cong \mathrm{Sp}_4(2). \end{aligned}$$

Note also that $P_j/O_2(P_j) \cong \mathrm{SL}_2(2)$ for $1 \leq j \leq 4$. By Lemma 3.4 (ii), $P_{23} = U_1 \cap U_2$ and $P = \langle P_1, P_2, P_3, P_4\rangle$.

Set $\mathcal{I} = \{1, 2, 3, 4\}$, and let

$$\mathcal{C} = (P/T, (P/P_k), k \in \mathcal{I})$$

be the corresponding chamber system. Thus $\mathcal{C}$ is an edge coloured graph with colours from $\mathcal{I} = \{1, 2, 3, 4\}$ and vertex set the right cosets P/T. Furthermore, two cosets Tg_1 and Tg_2 form a k-coloured edge if and only if $Tg_2g_1^{-1} \subseteq P_k$. Obviously P acts on $\mathcal{C}$ by multiplication of cosets on the right and this action preserves the coloured edges. For $\mathcal{J} \subseteq \mathcal{I}$, set $P_\mathcal{J} = \langle P_k \mid k \in \mathcal{J}\rangle$ and $\mathcal{C}_\mathcal{J} = (P_\mathcal{J}/T, (P_\mathcal{J}/P_k), k \in \mathcal{J})$. Then $\mathcal{C}_\mathcal{J}$ is the $\mathcal{J}$-connected component of $\mathcal{C}$ containing the vertex T.

We will show $\mathcal{C}$ locally resembles the corresponding chamber system in $\mathrm{F}_4(2)$. This means that for $\sigma \subset \mathcal{I}$ with $|\sigma| = 2$ we will show $P_\sigma/O_2(P_\sigma)$ is isomorphic to the corresponding group in $\mathrm{F}_4(2)$. Since $U_1/O_2(U_1) \cong \mathrm{Sp}_6(2)$ this is true if $\sigma \subseteq \{1, 2, 3\}$. Hence we may assume that $4 \in \sigma$. There are two possibilities for the relationship between P_2 and P_4 (they are both contained in U_2), but we may have $P_{24}/O_2(P_{24}) \cong \mathrm{SL}_3(2)$ or $P_{24} = P_2P_4$. We shall show that the latter is in fact the case. We will also prove $P_{14} = P_1P_4$. This is the purpose of the next lemma.

Lemma 3.5. *The subgroup $Z_2(T)$ is normalized by P_{14}, $P_{14} = P_1P_4$ and $P_{24} = P_2P_4$.*

Proof. Let $V = Z_2(T)$. Then, by Lemma 3.4 (iv), $V \cap Z(O_2(U_2)) \not\leq Z(O_2(U_1))$.

As $C_{O_2(U_1)/Z(O_2(U_1))}(T)$ has order 2 by Lemma 2.4 and $|V \cap Z(O_2(U_2))| = 2^3$ by Lemma 2.3, we deduce $V = (V \cap Z(O_2(U_1)))(V \cap Z(O_2(U_2)))$ has order 2^4 as $Z(T) = Z(O_2(U_1)) \cap Z(O_2(U_2))$.

Using Lemmas 2.3 and 2.4, $V \cap Z(O_2(U_1))$ and $VZ(O_2(U_1))$ are both normalized by P_1. Set

$$W = \langle V^{P_1} \rangle.$$

Then, as the set V^{P_1} has size at most 3, $W/(V \cap Z(O_2(U_1)))$ has order at most 2^3 and $W = V(W \cap Z(O_2(U_1)))$. Since $(W \cap Z(O_2(U_1)))/(V \cap Z(O_2(U_1)))$ has order at most 2^2, Lemma 2.3 implies $(W \cap Z(O_2(U_1)))/(V \cap Z(O_2(U_1)))$ is centralized by $O^2(P_1)$. But then $W/(V \cap Z(O_2(U_1)))$ is centralized by $O^2(P_1)$. Thus $W = V$. We may apply the same argument to U_2 to see that P_4 also normalizes V and so deduce that P_{14} acts on V which has order 2^4.

We have $[V, O_2(P_1)] \leq Z(O_2(U_1)) \cap Z(O_2(U_2)) = Z(T)$. Hence, as $[V, O_2(P_1)]$ is normalized by P_1, $[V, O_2(P_1)] = \langle x_1 \rangle$. Similarly $[V, O_2(P_4)] = \langle x_2 \rangle$. Therefore $O_2(P_1) \cap O_2(P_4)$ centralizes V and has index 4 in T. Thus $C_T(V) = O_2(P_1) \cap O_2(P_4)$. In particular, $O_2(P_1)$ acts as a transvection on V. Hence $C_V(O_2(P_1))$ has order 2^3 and so $C_V(O_2(P_1)) = V \cap Z(U_1)$ and $C_V(O_2(P_4)) = V \cap Z(O_2(U_2))$. Because $C_G(V) \leq U_1$, we have also shown $C_G(V) = O_2(P_1) \cap O_2(P_4)$.

Set

$$D = \langle O_2(P_1)^{N_G(V)}, O_2(P_4)^{N_G(V)} \rangle C_G(V)/C_G(V).$$

Then $D \cap U_1 = P_1$ and, as x_1 has at most 15 conjugates under the action of D, $|D| \leq 12 \cdot 15$. The structure of $\mathrm{Alt}(8) \cong \mathrm{GL}_4(2)$ therefore shows $D \cong \mathrm{SL}_2(2) \times \mathrm{SL}_2(2)$, or $\mathrm{O}_4^-(2) \cong \mathrm{Sym}(5)$.

Let $Q_{12} = O_2(P_{12})$, W_1 be the preimage of $C_{Z(O_2(U_1))/\langle x_1 \rangle}(Q_{12})$ and define $W = W_1V$. Then W is elementary abelian of order 2^5. Since $V = (V \cap Z(O_2(U_1)))(V \cap Z(O_2(U_2)))$,

$$\begin{aligned}
[W, Q_{12}] &= [W_1(V \cap Z(O_2(U_1)))(V \cap Z(O_2(U_2))), Q_{12}] \\
&\leq \langle r_1 \rangle [(V \cap Z(O_2(U_1)))(V \cap Z(O_2(U_2))), Q_{12}] \\
&= \langle x_1 \rangle [(V \cap Z(O_2(U_2))), Q_{12}] \\
&\leq \langle x_1 \rangle [(V \cap Z(O_2(U_2))), T] \\
&= \langle r_1 \rangle [(V \cap Z(O_2(U_2))), O_2(U_1)O_2(P_4)] \\
&= \langle x_1 \rangle [(V \cap Z(O_2(U_2))), O_2(U_1)] = \langle x_1 \rangle.
\end{aligned}$$

As $O_2(U_1)/Z(O_2(U_1))$ is a spin module for $\mathrm{Sp}_6(2)$,

$$C_{O_2(U_1))/Z(O_2(U_1))}(Q_{12}) = WZ(O_2(U_1))/Z(O_2(U_1))$$

by Lemma 2.4. We deduce that W is the preimage of $C_{O_2(U_1)/\langle x_1\rangle}(Q_{12})$ and thus W is normalized by P_{12}. Since $Z(O_2(U_1)) \cap Z(O_2(U_2)) = Z(T)$, we have $WZ(O_2(U_2))/Z(O_2(U_2))$ has order 2^2. It follows from Lemma 2.4 that $O^2(P_4)$ centralizes $WZ(O_2(U_2))/Z(O_2(U_2))$. Let $W_2 = \langle W^{P_4}\rangle$. Then $W_2 = W(W_2 \cap Z(O_2(U_2)))$. Since W/V has order 2, we infer that W_2/V has order at most 2^3. Thus $(W_2 \cap Z(O_2(U_2)))/(V \cap Z(O_2(U_2)))$ has order at most 2^2. It follows from Lemma 2.3 that $(W_2 \cap Z(O_2(U_2)))/(V \cap Z(O_2(U_2)))$ is centralized by $O^2(P_4)$. Therefore W/V is normalized by $TO^2(P_4) = P_4$. This shows that W is normalized by P_{124}. Notice that along the way we have shown that $P_{24} = P_2P_4$.

Suppose that $P_{14}/O_2(P_{14}) \cong \mathrm{O}_4^-(2)$. Then P_{14} acts irreducibly on V and so, as P_{12} does not normalize V, W is an irreducible P_{124}-module. As P_{14} has orbits of length 10 and 5 on V and $Z(T) \le V$, we have that P_{14} does not centralize any element in $W \setminus V$ and so P_{14} acts transitively on the 16 elements of $W \setminus V$. This means the orbits of P_{14} on the involutions of W have lengths 5, 10 and 16. Since 5 divides the order of D, we get that the number of conjugates of x_1 under P_{124} is divisible by 5 and, as $|x_1^{P_{12}}| = 10$, we conclude $|x_1^{P_{124}}| = 10$ or 15. But then $V = \langle x_1^{P_{124}}\rangle$, contradicting the fact that P_{124} acts irreducibly on W. Hence $P_{14}/O_2(P_{14}) \cong \mathrm{SL}_2(2) \times \mathrm{SL}_2(2)$ with $P_{14} = P_1P_4$ and this concludes the proof of the lemma. □

Proof of Theorem 3.3. Using Lemma 3.5 and the observations before the lemma yields that the chamber systems $\mathcal{C}_{1,2}$, $\mathcal{C}_{3,4}$ are projective planes, $\mathcal{C}_{2,3}$ is a generalized quadrangle and in both cases the parameters are $3,3$ and the remaining $\mathcal{C}_J$ with $|J| = 2$ are all complete bipartite graphs again with parameters $3,3$. Thus $\mathcal{C}$ is a chamber system of type F_4 (see [25]) in which all panels have 3 chambers. Since $U_1/O_2(U_1) \cong \mathrm{Sp}_6(2) \cong U_2/O_2(U_2)$, we have $\mathcal{C}_{1,2,3}$ and $\mathcal{C}_{2,3,4}$ are the $\mathrm{Sp}_6(2)$-building. Hence, as each connected rank 3 residue of $\mathcal{C}$ is a building of type C_3 and all the rank 2 residues of $\mathcal{C}$ are Moufang polygons, applying [25, Corollary 3] yields that the universal covering $\pi : \mathcal{C}' \longrightarrow \mathcal{C}$ has $\mathcal{C}'$ a building of type F_4 which also has three chambers on each panel. By [24, Proof of Theorem 10.2 on page 214] this building is uniquely determined by the two residues of rank three with connected diagram. Thus $\mathcal{C}'$ is isomorphic to the $\mathrm{F}_4(2)$ building and the type preserving automorphism group F of $\mathcal{C}'$ is isomorphic to $\mathrm{F}_4(2)$. Since $\mathcal{C}'$ is a 2-cover of $\mathcal{C}$, there is a subgroup U of F such that U contains U_1 and $U/D \cong P$ for a suitable normal subgroup D of U. As U_1 is isomorphic to a maximal parabolic subgroup of F, we deduce that $U = F$ and $D = 1$. Thus $P \cong F$. □

4. The structure of M

From now on we suppose that G is a group which satisfies the assumptions of Theorem 1.2. We set $M = N_G(Z)$. So $C_G(Z)$ has index at most 2 in M. Let $S \in \mathrm{Syl}_3(M)$ and $Q = F^*(M) = O_3(M)$.

Lemma 4.1. *We have $Z = Z(S) = Z(Q)$, $N_G(S) \le M$ and $S \in \mathrm{Syl}_3(G)$.*

Proof. Since $C_M(Q) \le F^*(Q) = Q$, we have that $Z = Z(Q) = Z(S)$. Therefore $N_G(S) \le N_G(Z) = M$ and, in particular, $S \in \mathrm{Syl}_3(N_G(S)) \subseteq \mathrm{Syl}_3(G)$. □

Let R^* be a normal subgroup of $C_G(Z)$ such that $R^*/Q \cong \mathrm{Q}_8 \times \mathrm{Q}_8$ and let $R \in \mathrm{Syl}_2(R^*)$. We have that M/Q embeds into $\mathrm{Out}(Q)$ and $\mathrm{Out}(Q)$ is isomorphic to $\mathrm{GSp}_4(3)$ by [11, III(13.7)]. We now locate M/Q in $\mathrm{Out}(Q)$. We will show that M/QR is isomorphic to $\mathrm{Sym}(3)$ or $2 \times \mathrm{Sym}(3)$. More precise information will be presented in Lemma 4.8. The next lemma provides our initial restriction on the structure of M.

Lemma 4.2. *We have that M/Q normalizes R^*/Q and is isomorphic to a subgroup of the subgroup $\mathbf{M}$ of $\mathrm{GSp}_4(3)$ which preserves a decomposition of the natural 4-dimensional symplectic space over $\mathrm{GF}(3)$ into a perpendicular sum of two non-degenerate 2-spaces. Furthermore, R/Q maps to $O_2(\mathbf{M})$.*

Proof. See [17, Lemma 3.1]. □

We next introduce a substantial amount of notation. We will use this for the remainder of the paper. We note now that the subgroups Q_1 and Q_2 defined below will be shown to have order 3^3 in Lemma 4.4.

Notation 4.3. (i) *Define R_1 and R_2 to be the two subgroups of R isomorphic to Q_8 which map to normal subgroups of $C_{\mathbf{M}}(Z(R)Q/Q)$.*

(ii) *For $i = 1, 2$, let $r_i \in Z(R_i)^\#$ and $K_i = C_G(r_i)$.*

(iii) *For $i = 1, 2$, define*

$$Q_i = [Q, R_i].$$

(iv) *For $i = 1, 2$, let $A_i \le Q_i$ be a fixed S-invariant subgroup of Q_i of order 3^2 and set $A = A_1 A_2$.*

(v) *For $i = 1, 2$, we let*

$$\langle \rho_i \rangle \le A_i$$

be such that $\langle \rho_i \rangle$ is inverted by r_i.

(vi) *Set $J = C_S(A)$ and $L = N_G(J)$.*

Most of this paper is devoted to the determination of K_1 and K_2. We will show that K_i is similar to a 2-centralizer in a group of type $\mathrm{F}_4(2)$ as defined in Definition 3.1 and, for $T \in \mathrm{Syl}_2(K_1)$, show that K_1, K_2 and T is an F_4 set-up. We then use Theorem 3.3 to obtain a subgroup $P \cong \mathrm{F}_4(2)$ of G. Our interim goal to achieve this objective is to show that $C_G(\rho_i)$ is isomorphic to the corresponding centralizer in $\mathrm{F}_4(2)$ or $\mathrm{Aut}(\mathrm{F}_4(2))$. We eventually do this in Lemma 8.2. However we begin more modestly by determining the precise structure of M.

Lemma 4.4. *The following hold.*

(i) $|S/Q| \le 3^2$.
(ii) $Q_1 = C_Q(r_2)$ *and* $Q_2 = C_Q(r_1)$ *and both are normal in* S*; and*
(iii) $Q_1 \cong Q_2 \cong 3^{1+2}_+$, $[Q_1, Q_2] = 1$ *and* $Q = Q_1Q_2$*;*
(iv) A *is elementary abelian of order* 3^3.

In particular, Q *has exponent* 3.

Proof. Part (i) follows from Lemma 4.2.

That Q_1 and Q_2 are normalized by S follows from the action of M on Q, as R_1Q/Q and R_2Q/Q are normalized by S/Q.

For $i = 1, 2$, we have that $C_Q(r_i)$ and $Q_i = [Q, r_i]$ commute by the Three Subgroup Lemma. Since Q_i has order 3^3 it follows that $Q_i \cong 3^{1+2}_+$. As r_1r_2 inverts Q/Z, r_2 inverts $C_{Q/Z}(r_1)$ and so $C_Q(r_1) = Q_2$ and $C_Q(r_2) = Q_1$. In particular, Q_1 and Q_2 commute and $Q = Q_1Q_2$. This proves (ii) and (iii). Finally (iv) follows from (ii) and (iii). $\square$

Lemma 4.5. *Every element of* Q *is* M*-conjugate to an element of* A.

Proof. It suffices to prove that every element of Q/Z is conjugate to an element of A/Z. Let $w \in Q/Z$. Then $w = x_1x_2$ where $x_i \in Q_i/Z$ by Lemma 4.4 (iii). Since, from the definition of A, for $i = 1, 2$, $(A \cap Q_i)/Z = A_i/Z$ has order 3 and R_i acts transitively on Q_i/Z, there exists $s_i \in R_i$ such that $w^{s_1s_2} = x_1^{s_1}x_2^{s_2} \in A/Z$. This proves the claim. $\square$

Recall that by hypothesis Z is not weakly closed in Q. Hence there is a $g \in G$ such that $Y = Z^g \le Q$ and $Y \ne Z$. We set

$$\begin{aligned} V &= ZY; \\ H &= \langle Q, Q^g\rangle; \text{ and} \\ W &= C_{Q^g}(Z)C_Q(Y). \end{aligned}$$

Notice that $C_Q(Y)$ normalizes $C_{Q^g}(Z)$ and so W is indeed a subgroup of G. Because of Lemma 4.5 we may and do suppose that $V \le A$. In particular, V is normalized by S. Before we continue our study of M, we investigate H.

Lemma 4.6. *The following statements hold.*

(i) *$S > Q$;*

(ii) *$Q \cap Q^g$ is elementary abelian of order 3^3 and is a normal subgroup of S;*

(iii) *$W = C_Q(Y)C_{Q^g}(Y)$ is a normal subgroup of H, $H/W \cong \mathrm{SL}_2(3)$, $WQ \in \mathrm{Syl}_3(H)$ and $W/(Q \cap Q^g)$ is a natural H/W module;*

(iv) *for $i = 1, 2$, $V \cap Q_i = Z$ and $A \neq Q \cap Q^g$;*

(v) *$A = [Q, W] \leq W$, $A/Z = C_{Q/Z}(S) = C_{Q/Z}(W)$ and A is normal in $N_G(S)$; and*

(vi) *for $i = 1, 2$, $[WQ/Q, R_iQ/Q] \neq 1$.*

Proof. As Q is extraspecial, $C_Q(Y)$ is non-abelian of order 3^4. By Lemma 4.1, M^g/Q^g has Sylow 3-subgroups of order at most 9 and $C_Q(Y) \leq M^g$ so we have $Z = C_Q(Y)' \leq Q^g$. In particular we now have $S > Q$ for else $C_Q(Y) \leq Q^g$ and then $Z = C_Q(Y)' \leq (Q^g)' = Y$ which is a contradiction. In particular, (i) holds.

Since $\Phi(Q \cap Q^g) \leq Z \cap Y = 1$, $Q \cap Q^g$ is elementary abelian.

Because $V \leq Q \cap Q^g$, we have $[V, Q] = Z$ and $[V, Q^g] = Y$ and so H normalizes and acts non-trivially on V with $H/C_H(V) \cong \mathrm{SL}_2(3)$.

Turning our attention to W, we have

$$[W, Q] = [C_Q(Y)C_{Q^g}(Z), Q] = Z[C_{Q^g}(Z), Q].$$

Since $[[C_{Q^g}(Z), Y], Q] = 1 = [Q, Y, C_{Q^g}(Z)]$, the Three Subgroup Lemma implies that $[C_{Q^g}(Z), Q] \leq C_Q(Y) \leq W$. Therefore

$$[Q, W] \leq C_Q(Y) \leq W$$

and, similarly, $[W, Q^g] \leq C_{Q^g}(Z) \leq W$. Hence H normalizes W and of course $W \leq C_G(V)$.

As $[C_H(V), Q] \leq C_Q(V) = C_Q(Y) \leq W$, H/W is a central extension of $\mathrm{SL}_2(3)$. Since H acts transitively on the four subgroups of order 3 in V, and each such subgroup determines uniquely a subgroup of H we have that Q^H has exactly 4 members. Now $O^3(H)W/W$ is a central extension of a nilpotent group and is thus nilpotent. Let T be a Sylow 2-subgroup of $O^3(H)$. Then as $O^3(H)W/W$ is nilpotent, Q normalizes and does not centralize TW/W. It follows that $H = WTQ$ and then the action of Q on TW/W and the fact that $T/C_T(V) \cong \mathrm{Q}_8$ imply that $T \cong \mathrm{Q}_8$ and that $H/W \cong \mathrm{SL}_2(3)$, as by [11, Satz V.25.3] the Schur multiplier of a quaternion group is trivial.

Using that $O^3(H)$ acts transitively on $V^\#$, we see that $O^3(H)$ does not normalize any non-trivial subgroup of $(W \cap Q)/(Q \cap Q^g)$.

Assume $Q \cap Q^g = V$. Then $|W| = 3^6$. As $W' \leq V$, W is generated by groups of exponent 3 and W is non-abelian, we have $\Phi(W) = V$.

Let $f \in H$ be an involution. Then $fW \in Z(H/W)$ and, by Burnside's Lemma, f does not centralize $W/\Phi(W)$ and neither does it invert $W/\Phi(W)$, for then, as f inverts V, W would be abelian. Therefore, setting $W_0 = C_W(f)V$, we have $W_0 > V$. Then, as the faithful representations of $\mathrm{SL}_2(3)$ in characteristic 3 have even dimension and the minimal faithful representation for $\mathrm{PSL}_2(3)$ is 3, $|W_0/V| = 3^2$ and W_0 is centralized by $O^3(H)$ and normalized by Q; in particular, $Q \cap W_0 \le V$ by the comments at the end of the last paragraph. But then $(W \cap Q)W_0 = W_0(W \cap Q^g) = W$ which means that

$$[W, Q] = [W_0, Q][W \cap Q, Q] \le V.$$

Consequently $O^3(H)$ centralizes W/V which is a contradiction, as we have already remarked that f does not centralize W/V. Therefore $Q \cap Q^g > V$.

Since $Q \cap Q^g$ is abelian and Q is extraspecial of order 3^5, we now have that $|Q \cap Q^g| = 3^3$ and $W/(Q \cap Q^g)$ is a natural $\mathrm{SL}_2(3)$-module. This completes the proof of the first two statements in (ii) and all of (iii).

Since H acts 2-transitively on the non-trivial cyclic subgroups of V, $N_G(V) = (N_M(V) \cap N_{M^g}(V))H$ and therefore $N_G(V)$ normalizes $Q \cap Q^g$. From the choice of $V \le A$, we have $S \le N_G(V)$. This is the last statement in (ii).

Suppose that $V \le Q_i$ for some $i \in \{1, 2\}$. Then $C_M(V) \ge R_{3-i}$ and so R_{3-i} acts on $Q \cap Q^g$. Since $|Q \cap Q^g : V| = 3$, we obtain $Q \cap Q^g \le C_Q(r_{3-i}) = Q_i$ contrary to $Q \cap Q^g$ being elementary abelian of order 3^3. Hence V is not contained in Q_i for $i = 1, 2$. If $A = Q \cap Q^g$, then

$$Y = [A, C_{Q^g}(Z)] \le [A, S] = Z,$$

which is impossible. Hence we also know that $A \ne Q \cap Q^g$. Thus (iv) holds.

If $[Q_1, W] \le Z$, then $[Q, W] = [Q_1, W][Q_2, W] \le A_2$. Therefore using (iv),

$$[C_Q(V), W] = [C_Q(V), C_{Q^g}(V)]Z \le Q \cap Q^g \cap A_2 = Z.$$

Since $|Q \cap Q^g| = 3^3$ by (ii), $Y = [Q \cap Q^g, C_{Q^g}(V)] \le [Q, W] = Z$ which is impossible. Thus $[Q_1, W] = A_1$ and similarly $[Q_2, W] = A_2$. Now $[Q, W] = A$ and consequently $[Q, S] = A$. This proves (v).

Finally, suppose that $[WQ, R_1Q] \le Q$. Then $[Q_1, W] \le A_1$ and is R_1-invariant. Hence $[Q_1, W] \le Z$ and this contradicts (v). Thus $[WQ, R_1Q] \not\le Q$ and (vi) holds. □

Now we are in a position to determine M. For this set

$$M_0 = RQ$$

and let f be an involution in H. Then f inverts V and thus $f \in M$. We refine our choice of R so that $R\langle f\rangle$ is a Sylow 2-subgroup of $M_0S\langle f\rangle$.

Lemma 4.7. *We have that Z is the unique G-conjugate of Z in both Q_1 and Q_2.*

Proof. Suppose that $g \in G$, $Z^g \le Q_1$ with $Z^g \ne 1$. Then, using Z^g in place of Y, Lemma 4.6 (iv) applies to give a contradiction. □

Lemma 4.8. *The following hold.*

(i) *$S = WQ$ and $|S/Q| = 3$; and*

(ii) *One of the following holds:*

(a) *$M = M_0S\langle f\rangle$, $C_M(Z) = M_0S$ and $M/M_0 \cong \mathrm{Sym}(3)$; or*

(b) *$|M : M_0S\langle f\rangle| = 2$, $C_M(Z) = M_0S\langle t\rangle$ where t is an involution which exchanges R_1 and R_2, centralizes V and inverts SM_0/M_0 and $M/M_0 = \langle t, f\rangle SM_0/M_0 \cong 2 \times \mathrm{Sym}(3)$ with centre $\langle tf\rangle M_0/M_0$.*

Proof. We have seen in Lemma 4.6 (i) and (v) that $|S/Q| \ge 3$ and $A/Z = C_{Q/Z}(S) = C_{Q/Z}(W)$.

Suppose that $|S/Q| = 3^2$ and assume that B is an abelian subgroup of Q which is normal in S of order 3^3 with $B \ne A$. For $i = 1, 2$, let $s_i \in S$ be such that $[s_i, R_{3-i}] \le Q$. Then $[B, s_i] \le B \cap A \cap Q_i \le A_i$. Thus if s_i does not centralizes B/Z, then $A_i \le B$. Since $S = Q\langle s_1, s_2\rangle$ and $B \ne A$, without loss of generality we may suppose that $A_1 \le B$ and $[B, s_2] \le Z$. In particular, $B \le Q_1A$ as $C_{Q/Z}(s_2) = Q_1A/Z$. But then A_1 is centralized by $AB = Q_1A$ and we have a contradiction as $Z(Q_1A) = A_2$. Thus, if $B \le Q$ is a normal abelian subgroup of S of order 3^3, then $B = A$. Taking $B = Q \cap Q^g$, we now have that $Q \cap Q^g = A$ a possibility which is eliminated by Lemma 4.6 (iv). Thus $|S/Q| = 3$. This proves (i).

We know that f inverts $W/(Q \cap Q^g)$ and so WQ/Q is inverted by f. In particular, $M_0S\langle f\rangle/M_0 \cong \mathrm{Sym}(3)$. If $M = M_0S\langle f\rangle$, then (ii)(a) holds. So assume that $M > M_0S\langle f\rangle$. As M inverts Z, we have $M = C_M(Z)\langle f\rangle$. Since, by Lemma 4.2, $C_M(Z)/Q$ is isomorphic to a subgroup of $\mathrm{Sp}_2(3) \wr 2$ and since S/Q has order 3, Lemma 4.6 (vi) implies that $C_M(Z)/M_0 \cong 3 \times 2$ or $\mathrm{Sym}(3)$. Especially, there is a 2-element $t \in C_M(Z) \setminus M_0$ which normalizes $R\langle f\rangle$ and swaps R_1 and R_2. Because $R\langle t\rangle$ is isomorphic to a Sylow 2-subgroup of $\mathrm{Sp}_2(3) \wr 2$, we may as well assume that t is an involution and that t normalizes S.

Since t normalizes S and swaps R_1 and R_2, t also interchanges Q_1 and Q_2 and normalizes A. It follows that t normalizes V. Without loss of generality we may now additionally assume that t normalizes Y. Thus t normalizes $Q \cap Q^g$ as well as A. Since t centralizes Z, $[Q, t]$ is extraspecial of order 3^{1+2}. Hence either t centralizes V and $Q/C_Q(V)$ or t inverts V/Z and $Q/C_Q(V)$. Multiplying t by r_1r_2, we may assume that t centralizes V. If S/Q is centralized by t, we now have $S/C_Q(V)$ is centralized by t. However, as $[Q,S](Q\cap Q^g) = C_Q(V)/(Q\cap Q^g)$, we see that $S/(Q\cap Q^g)$ is extraspecial and since t centralizes $S/C_Q(V)$, Burnside's Lemma implies that t centralizes $S/(Q\cap Q^g)$. Then t also centralizes Q which is a contradiction. Hence t inverts S/Q and therefore $C_M(Z)/M_0$ has the structure described in (ii)(b). $\square$

5. THE STRUCTURE OF $L = N_G(J)$

In this section we continue to use the notation introduced in 4.3. We also recall $H = \langle Q, Q^g\rangle$ and f is an involution in $H\cap M$ which inverts Z.

We will show that J is the Thompson subgroup of S and determine $L = N_G(J)$.

Set

$$H_1 = H^{r_1}, W_1 = W^{r_1} \text{ and } V_1 = V^{r_1}.$$

Lemma 5.1. *We have $W \neq W_1$ and $H \neq H_1$.*

Proof. Notice that r_1 inverts A_1/Z and centralizes A_2/Z. Therefore, $V^{r_1} \neq V$. Since

$$W' = [C_Q(V), C_{Q^g}(V)]V \le Q\cap Q^g \cap [Q,W] = Q\cap Q^g\cap A = V,$$

we see $W' = V$ and $W_1' = V_1$. Thus W and W_1 are not equal and so also $H \neq H_1$. $\square$

Lemma 5.2. *For $i = 1, 2$, we have ρ_i is not G-conjugate to an element of Z. In particular, A contains exactly seven G-conjugates of Z.*

Proof. By definition $\langle\rho_i\rangle \le Q_i$ for $i = 1, 2$. Hence Lemma 4.7 gives $\langle\rho_i\rangle$ is not a G-conjugate of Z.

Since $V\cup V_1 \subset A$, we now see A contains exactly seven G-conjugates of Z, three Q-conjugates of $\langle\rho_1\rangle$, and three Q-conjugates of $\langle\rho_2\rangle$. $\square$

We can now describe the structure of L.

Lemma 5.3. *The following hold.*

(i) *$J = J(S)$ is elementary abelian of order 3^4.*
(ii) *L controls G-fusion of elements of J.*
(iii) *$J = C_G(J)$.*
(iv) *L preserves a quadratic form q of $+$-type on J up to similarity.*
(v) *Set $L_* = \langle H, H_1, r_1, r_2\rangle$. Then $L_*/J \cong \mathrm{GO}_4^+(3)$ and either*
  (a) *if $M = M_0S\langle f\rangle$, then $L = L_*$; or*

(b) *if* $M > M_0S\langle f\rangle$, *then* $L/J \cong \mathrm{CO}_4^+(3)$. *(Here* $\mathrm{CO}_4^+(3)$ *is the group which preserves* q *up to similarity.)*

Proof. By construction A is elementary abelian and so $A \le C_Q(V) \le W$ and $A \le C_Q(V_1) \le W_1$. Since S centralizes A/Z and since in $\mathrm{GL}_3(3)$ such a centralizer has order 18, we infer that $J = C_S(A)$ has order 3^4. Since A has index 3 in J, J is abelian. Suppose that B is an abelian subgroup of S of order at least 3^4. We may assume that $B \ge Z$. Thus by Lemma 4.8, $B \cap Q$ is an abelian subgroup of Q of order at least 3^3 and hence of order exactly 3^3. Using that $(B\cap Q)/Z$ is centralized by $QB = S$, Lemma 4.6 (iii) yields $B \cap Q = A$. But then $B \le C_S(A) = J$ and we have $B = J$. Hence $J = J(S)$ is the Thompson subgroup of S. Since J centralizes V, $J \le S \cap C_G(V) = W$. Thus $J = J(W)$ and similarly $J = J(W_1)$. In particular, $L \ge \langle H, H_1\rangle N_G(S)$. Since J contains A, if J is not elementary abelian, then $\Phi(J) = Z$. But then Z is normalized by H, which is a contradiction as H acts irreducibly on V. Thus J is elementary abelian. This proves (i). Part (ii) follows from [1, 37.6] as J is abelian.

We have that $C_G(J) \le C_G(Z) < M$. Since J acts non-trivially on both R_1Q/Q and R_2Q/Q, and JM_0/M_0 is inverted by t when $M > M_0S\langle f\rangle$ (see Lemma 4.8 (ii)), we have $C_M(J) \le S\langle r_1, r_2\rangle$. Since r_1Q and r_2Q act non-trivially on A/Z, we have $C_G(J) \le S$. Hence $J \le C_G(J) = C_S(J) \le C_S(A) \le J$ and this proves (iii).

Define

$$\mathcal{S}(J) = \{j \in J^{\#} \mid j^l \in Z \text{ for some } l \in L\}.$$

Consider $S/J = Q_1Q_2J/J$. Then $S/J \in \mathrm{Syl}_3(L_*/J) \subseteq \mathrm{Syl}_3(L/J)$. We have $[J, Q_1] = A_1 = C_J(Q_2)$ and $[J, Q_2] = A_2 = C_J(Q_1)$. In addition, $[J, S] = [J, Q] = [W, Q] = A$ and $C_J(S) = Z$.

Now $\langle Z^{L_*}\rangle \ge \langle Z^H\rangle\langle Z^{H_1}\rangle = VV_1 = A$ and, as $A \not\le Q \cap Q^g$, A is not normalized by H. Hence $\langle Z^{L_*}\rangle = J$ and, in particular, L_* and, consequently, L acts irreducibly on J. Thus there are members of $\mathcal{S}(J)$ in $J\backslash A$. By Lemma 5.2 there are exactly 14 elements of $\mathcal{S}(J)$ in A and in $J\backslash A$ there are a multiple of 18 such elements. Thence $|\mathcal{S}(J)| = 14 + n\cdot 18$ for some integer $n \ge 1$. Since $|J| = 3^4$, using the fact that $|\mathcal{S}(J)|$ divides $|\mathrm{GL}_4(3)|$ we infer that $|\mathcal{S}(J)| = 32$.

Using Lemma 2.12 with $\langle a\rangle = Q_1J/J$ and $\langle b\rangle = Q_2J/J$, yields that S preserves a quadratic form with any element of $\mathcal{S}(J)$ as a singular vector. Since S/J contains W_1/J and W_2/J which both act quadratically on J with $[J, W] = [J, J(Q\cap Q^g)] = [J, (Q \cap Q^g)] = V$ and $[J, W] = [J, W]^{r_1} = V_1$ we see that for any such form V and V_1 would consist of singular vectors. It follows that $\mathcal{S}(J)$ is the set of singular vector of a $+$-type quadratic form on J. Since this set is by design invariant under the action of L, we have L/J is isomorphic to a subgroup of $\mathrm{CO}_4^+(3)$ by Lemma 2.11. Thus (iv) is true. Now HH_1 contains $S = WW_1$ which is a Sylow 3-subgroup of G, H acts irreducibly on V and H_1 acts irreducibly on V_1, it follows that $HH_1/J \cong \Omega_4^+(3)$. Conjugation by r_1 exchanges H and H_1, $\langle r_1r_2\rangle H_1/W_1 \cong \mathrm{GL}_2(3)$ and so we infer that $L_*/J \cong \mathrm{GO}_4^+(3)$ and L_* is normal in L. By the Frattini Argument, $L = N_L(S)L_* = N_M(S)L_*$ and so (v) holds. □

Lemma 5.4. *We have ρ_1 is G-conjugate to ρ_2 if and only if $SR\langle f\rangle$ has index 2 in M.*

Proof. This is a consequence of Lemma 5.3(ii) and (v). □

Recall the notation introduced in 2.13 and 2.15.

Lemma 5.5. *The sets $\mathcal{P}(J)$ and $\mathcal{M}(J)$ are fused in L if $L > L_*$ and we have $|\mathcal{S}(J)| = 16$, $|\mathcal{P}(J)| = |\mathcal{M}(J)| = 12$.*

Proof. This follows directly from Lemma 5.3. □

Lemma 5.6. *For $i = 1, 2$, $C_L(r_i) = C_{L_*}(r_i)$, $[J, r_i] = \langle\rho_i\rangle$, $|C_J(r_i)| = 3^3$ and $C_L(r_i)/C_J(r_i)\langle r_i\rangle \cong \mathrm{GO}_3(3) \cong 2 \times \mathrm{Sym}(4)$.*

Proof. We have that $|C_S(r_i)| = 3^4$ and r_i inverts Q_iJ/J. Hence $|C_J(r_i)| = 3^3$. It follows that both r_1 and r_2 are reflections on J. If $L > L_*$, then $r_1^t = r_2$ and so $C_L(r_i) = C_{L_*}(r_i)$. Since r_1 and r_2 are reflections and since $L_*/J \cong \mathrm{GO}_4^+(3)$ by Lemma 5.3, we have $C_L(r_i)/C_J(r_i)\langle r_i\rangle \cong \mathrm{GO}_3(3) \cong 2 \times \mathrm{Sym}(4)$. □

From Lemma 5.6 we have $[J, r_1] = \langle\rho_1\rangle$ and $[J, r_2] = \langle\rho_2\rangle$ are non-singular 1-dimensional spaces in J. We fix notation so that $\langle\rho_1\rangle \in \mathcal{P}(J)$ and $\langle\rho_2\rangle \in \mathcal{M}(J)$.

Lemma 5.7. *The following hold:*

(i) *V and V_1 are of Type S;*
(ii) *A_1 is of Type D+;*
(iii) *A_2 is of Type D-;*
(iv) *$\langle\rho_1, \rho_2\rangle$ is of type N+;*
(v) *$|\mathcal{S}(C_J(r_1))| = 4$, $|\mathcal{M}(C_J(r_1))| = 6$ and $|\mathcal{P}(C_J(r_1))| = 3$; and*
(vi) *$|\mathcal{S}(C_J(r_2))| = 4$, $|\mathcal{M}(C_J(r_2))| = 3$ and $|\mathcal{P}(C_J(r_2))| = 6$.*

Proof. Parts (i)–(iv) are obvious. By Lemma 5.6 we have that $|C_J(r_i)| = 3^3$ for $i = 1, 2$. Since J is a quadratic space of plus type, it follows that $C_J(r_1)$ has an orthonormal basis consisting of members of $\mathcal{P}(J)$ and $C_J(r_2)$ has an orthonormal basis consisting of elements of $\mathcal{M}(J)$. Thus (v) and (vi) hold. □

Lemma 5.8. *If $\widetilde{\rho}_i \in C_J(r_i)$ is L_*-conjugate to ρ_i, then $\langle \rho_i, \widetilde{\rho}_i \rangle$ has Type N-. In particular, $|\mathcal{P}(\langle \rho_i, \widetilde{\rho}_i \rangle)| = |\mathcal{M}(\langle \rho_i, \widetilde{\rho}_i \rangle)| = 2$.*

Proof. Suppose that $\widetilde{\rho}_i \in C_J(r_i)$ is L_*-conjugate to $\langle \rho_i \rangle$. Then, as $\langle \rho_i \rangle = [J, r_i]$, ρ_i is perpendicular to $C_J(r_i)$. It follows that $\widetilde{\rho}_i$ is perpendicular to ρ_i and this means that $\langle \rho_i, \widetilde{\rho}_i \rangle$ is of Type N-. □

6. Two 3-centralizers

In this section we determine the structure of $C_G(\rho_1)$ and $C_G(\rho_2)$. We first show that these centralisers do not have non trivial normal $3'$-subgroups. Recall the notation of 4.3 and that $f \in M$ is an involution inverting Z.

Lemma 6.1. *J does not normalize any non-trivial $3'$-subgroups.*

Proof. Suppose that Y is a non-trivial $3'$-subgroup normalized by J. Then, as every subgroup of J of order 27 contains a conjugate of Z by Lemma 2.14, we may assume that $X = C_Y(Z) \neq 1$. As X is normalized by $A = J \cap Q$ and X normalizes Q, $[A, X] \leq Q \cap X = 1$ and hence $X \leq C_M(A) = J$ as A is a maximal abelian subgroup of Q. But then $X = 1$ which is a contradiction. This proves the lemma. □

Lemma 6.2. *For $i = 1, 2$, $C_M(\rho_i) = Q_{3-i}R_{3-i}J\langle fr_i \rangle$ and $C_{C_M(Z)}(\rho_i)/\langle \rho_i \rangle$ is isomorphic to the centralizer of a non-trivial 3-central element in $\mathrm{PSp}_4(3)$ and Z is inverted in $C_M(\rho_i)$.*

Proof. Since $\rho_i \in A_i \leq J$ and since $[Q_1, Q_2] = 1$ and $[Q_i, R_{3-i}] = 1$, we certainly have $C_M(\rho_i) \geq Q_{3-i}R_{3-i}J$. Furthermore, f inverts J and so f inverts ρ_i and as r_i also inverts ρ_i, we have $C_M(\rho_i) \geq Q_{3-i}R_{3-i}J\langle fr_i \rangle$ which has index either 24 or 48 in M dependent upon whether or not $M = RS\langle f \rangle$ respectively. Since Q_i contains twelve Q-conjugates of $\langle \rho_i \rangle$, Lemma 5.4 implies $C_M(\rho_i) \geq Q_{3-i}R_{3-i}J\langle fr_i \rangle$.

Because r_if inverts Z, we have $C_{C_M(Z)}(\rho_i)/\langle \rho_i \rangle = Q_{3-i}R_{3-i}J/\langle \rho_i \rangle$ with R_{3-i} acting faithfully on Q_{3-i}. Thus the final statement also is valid by Lemma 2.10. □

In the next two lemmas we pin down two possible structures of $C_G(\rho_1)$ and $C_G(\rho_2)$. In fact in $\mathrm{F}_4(2)$ we have that both are isomorphic to $3 \times \mathrm{Sp}_6(2)$. That this is the case in our group will be proved later in Lemma 8.2.

Lemma 6.3. *For $i = 1, 2$ either $C_G(\rho_i) \cong 3 \times \mathrm{Aut}(\mathrm{SU}_4(2))$ or $C_G(\rho_i) \cong 3 \times \mathrm{Sp}_6(2)$. Furthermore, r_i inverts ρ_i and centralizes $C_G(\rho_i)/\langle \rho_i \rangle$.*

Proof. We consider $C_G(\rho_i)/\langle\rho_i\rangle$. By Lemma 6.2, $C_{C_{M(Z)}}(\rho_i)/\langle\rho_i\rangle$ is isomorphic to a 3-centralizer in $\mathrm{PSp}_4(3)$. Since $J/\langle\rho_i\rangle$ normalizes no non-trivial $3'$-subgroup of $C_G(\rho_i)$ by Lemma 6.1 and Z is inverted by fr_i, we may apply Theorem 2.9 to obtain $C_G(\rho_i)/\langle\rho_i\rangle \cong \mathrm{Aut}(\mathrm{SU}_4(2))$ or $\mathrm{Sp}_6(2)$ or that $C_G(\rho_i) = C_M(\rho_i)$. The latter possibility is dismissed as $C_L(\rho_i)$ has index 2 in $\langle\rho_i\rangle C_{L_*}(r_i)$ and so, by Lemma 5.6,

$$C_L(\rho_i) \cong 3 \times 3^3 : (2 \times \mathrm{Sym}(4))$$

does not normalize Z.

The Sylow 3-subgroup of $C_G(\rho_i)$ is $\langle\rho_i\rangle \times Q_{3-i}C_J(r_i)$ and hence the extension $C_G(\rho_i)/\langle\rho_i\rangle$ splits by Gaschütz Theorem. Finally we have that r_i centralizes $Q_{3-i}J/\langle\rho_i\rangle$ and, as no automorphism of either $\mathrm{Aut}(\mathrm{SU}_4(2))$ or $\mathrm{Sp}_6(2)$ of order 2 centralizes such a subgroup, we infer that r_i centralizes $C_G(\rho_i)/\langle\rho_i\rangle$ and of course we also know that ρ_i is inverted by r_i. □

Lemma 6.4. *We have $C_G(\rho_1) \cong C_G(\rho_2)$.*

Proof. By Lemma 6.3, $C_G(\rho_1)/\langle\rho_1\rangle \cong \mathrm{Sp}_6(2)$ or $\mathrm{Aut}(\mathrm{SU}_4(2))$.

Assume that $C_G(\rho_1)/\langle\rho_1\rangle \cong \mathrm{Sp}_6(2)$. Using Lemma 5.7 (v), we have some $\widetilde{\rho_1} \in \mathcal{P}(C_J(\rho_1))$ and as $|\mathcal{P}(C_J(\rho_1))| = 3$, $C_{E(C_G(\rho_1))}(\widetilde{\rho_1}) \cong 3 \times \mathrm{Sp}_4(2)$ from the structure of $\mathrm{Sp}_6(2)$. Therefore $E(C_G(\langle\rho_1, \widetilde{\rho_1}\rangle)) \cong \mathrm{Sp}_4(2)'$. Lemma 5.8, yields that $\mathrm{Sp}_4(2)'$ is involved in the centralizer of a 3-element in $C_G(\rho_2)$. As there are no such 3-elements in $\mathrm{SU}_4(2)$ [4], Lemma 6.3 implies $E(C_G(\rho_2))/\langle\rho_2\rangle \cong \mathrm{Sp}_6(2)$. Hence Lemma 6.4 holds. □

7. Building a signalizer in the centralizers of r_1 and r_2

In this section we begin the construction $K_i = C_G(r_i)$ for $i = 1, 2$. We give a brief overview of our plans for $i = 1$ to guide the reader through the technicalities involved. Our final aim is to show that K_1 is similar to a 2-centralizer in a group of type $\mathrm{F}_4(2)$ (see Definition 3.1). Hence we aim to show that K_1 is an extension of a 2-group by $\mathrm{Sp}_6(2)$. Further we show this 2-group is a product of an extraspecial group of order 2^9 by an elementary abelian group. Our first aim is to construct the extraspecial group Σ_1, and show that it is normalized by $C_L(r_1)$. Note that $C_J(r_1) \le C_L(r_1)$ and the former group is elementary abelian of order 3^3.

We briefly consider the situation in our target group. In $\mathrm{F}_4(2)$ there are exactly four maximal subgroups of $C_J(r_1)$ with centralizers in Σ_1 which properly contain $\langle r_1\rangle$ and these maximal subgroups centralize a quaternion group of order eight in Σ_1. In our group G, the first problem is to find these quaternion groups. For this we pick a set of four maximal subgroups of $C_J(r_1)$, which are conjugate to A_2. They all contain a conjugate of ρ_2. By Lemma 6.3 there are exactly two possibilities for the structure of $C_G(\rho_2)$. Examining these structures shows $C_{C_G(\rho_2)}(A_2)/\langle\rho_2\rangle \cong 3_+^{1+2}{:}\mathrm{SL}_2(3)$. Hence $C_{C_G(\rho_2)\cap C_G(r_1)}(A_2)/\langle\rho_2\rangle \cong \mathrm{SL}_2(3)$. This shows that $O_2(C_{C_G(\rho_2)\cap C_G(r_1)}(A_2)) \cong \mathrm{Q}_8$, and this is one of the quaternion groups we are looking for. As A_2 has four conjugates under $C_L(r_1)$, we now get a set of four quaternion groups. The problem is now to show these four quaternion groups generate a 2-group Σ_1 which is extraspecial of order 2^9. This will be done in Lemma 7.12. Furthermore, the very construction guarantees that $C_L(r_1)$ acts on Σ_1.

We continue to use the notation from 2.13, 2.15 and 4.3. Additionally we introduce

Notation 7.1. *For $i = 1, 2$, $I_i = C_J(r_i)$ and $F_i = C_L(r_i)$.*

Notice that by Lemma 5.6, F_i acts on I_i and $F_i/I_i\langle r_i\rangle \cong 2\times\mathrm{Sym}(4)$. As explained above we intend to determine a large signalizer for I_i (a $3'$-group which is normalized by I_i). We begin with two easy observations.

Lemma 7.2. *For $i = 1, 2$, $C_{C_M(Z)}(r_i) = Q_{3-i}R_1R_2I_i$ and $C_S(r_i) = Q_{3-i}I_i \in \mathrm{Syl}_3(C_M(r_i)) \subseteq \mathrm{Syl}_3(K_i)$.*

Proof. Obviously $C_{C_M(Z)}(r_i) \geq Q_{3-i}R_1R_2C_J(r_i)$ and so Lemma 4.8 (ii) yields equality. Therefore, $C_S(r_i) = Q_{3-i}I_i \in \mathrm{Syl}_3(C_M(r_i))$ and $Z(C_S(r_i)) = Z$. Thus $N_{K_i}(C_S(r_i)) \leq N_G(Z) = M$. In particular, $C_S(r_i) \in \mathrm{Syl}_3(K_i)$. □

Lemma 7.3. *We have r_1 is G-conjugate to r_2 if and only if r_1 is M-conjugate to r_2.*

Proof. Obviously if r_1 and r_2 are conjugate in M then they are conjugate in G. Suppose then that $r_1 = r_2^g$ for some $g \in G$. By Lemma 7.2, for $i = 1, 2$, $C_S(r_i) \in \mathrm{Syl}_3(C_G(r_i))$ and $Z = Z(C_S(r_i))$. Since $r_1 = r_2^g$, $C_S(r_2)^g \in \mathrm{Syl}_3(C_G(r_1))$. Thus there is $h \in C_G(r_1)$ such that $C_S(r_2)^{gh} = C_S(r_1)$. But then

$$Z^{gh} = Z(C_S(r_2))^{gh} = Z(C_S(r_1)) = Z$$

which means that $gh \in M$. Hence r_1 and r_2 are M-conjugate. □

Recall, for $i = 1, 2$,

$$I_i = C_J(r_i) = J \cap E(C_G(\rho_i))$$

as, by Lemma 6.3, $E(C_G(\rho_i)) = C_{C_G(\rho_i)}(r_i)$.

Lemma 7.4. *Suppose that $\widetilde{\rho}_1 \in \mathcal{P}(I_1)$ and $\widetilde{\rho}_2 \in \mathcal{M}(I_2)$. Then, for $i = 1, 2$, in $E(C_G(\widetilde{\rho}_i))\langle r_i\rangle$, r_i is an involution which has $\mathrm{Sp}_4(2)'$ as* *a composition factor of its centralizer. Moreover, $I_i \cap E(C_G(\widetilde{\rho_i}))$ is of Type N-.*

Proof. For $i = 1, 2$, the definition of I_i, yields $r_i \in C_G(\widetilde{\rho_i})$. Now r_i normalizes $E(C_G(\widetilde{\rho_i}))$ and centralizes $I_i \cap E(C_G(\widetilde{\rho_i}))$ which has order 9.

On the other hand, in $C_G(\rho_i)$, as there are only three conjugates of $\langle\widetilde{\rho_i}\rangle$ in I_i by Lemma 5.7(v) and (vi), we have that

$$C_{E(C_G(\rho_i))}(\widetilde{\rho_i}) \approx 3 \times 3^2.\mathrm{Dih}(8)$$

if $E(C_G(\rho_i)) \cong \mathrm{SU}_4(2)$ and

$$C_{E(C_G(\rho_i))}(\widetilde{\rho_i}) \approx 3 \times \mathrm{Sp}_4(2)$$

if $E(C_G(\rho_i)) \cong \mathrm{Sp}_6(2)$. As $I_i \le E(C_G(\rho_i))$, it follows that

$$I_i \cap [I_i, C_{E(C_G(\rho_i))}(\widetilde{\rho_i})]$$

is of Type N-. Now deploying Lemmas 2.2 and 2.5 (ii), $C_{E(C_G(\widetilde{\rho_i}))}(r_i) \cong \mathrm{Sp}_4(2)$ if $E(C_G(\widetilde{\rho_i})) \cong \mathrm{SU}_4(2)$ and has shape $2^5.\mathrm{Sp}_4(2)$ when $E(C_G(\widetilde{\rho_i})) \cong \mathrm{Sp}_6(2)$. In particular, the main claim in the lemma is true. We have already observed that $I_i \cap [I_i, C_{E(C_G(\rho_i))}(\widetilde{\rho_i})]$ has Type N- and as this group is $I_i \cap E(C_G(\widetilde{\rho_i}))$ we have the last part of the lemma. □

We can now locate the four maximal subgroups of I_i, whose centralizers contain the quaternion groups we are looking for. Recall that, for $i = 1, 2$, $A_{3-i} = A \cap Q_{3-i}$ is a hyperplane of I_i which with respect to the quadratic form on J is a degenerate 2-dimensional subspace which contains one conjugate of Z and three conjugates of $\langle\rho_i\rangle$. Therefore A_1 has Type D+ and has A_2 Type D- in the sense of Notation 2.15. Consequently the set $A_{3-i}^{F_i}$ has order 4. We let the four F_i-conjugates of A_{3-i} be $I_i^1 = A_{3-i}$, I_i^2, I_i^3 and I_i^4. Then, for $1 \le j < k \le 4$, we have $I_i^j \cap I_i^k$ is an M-conjugate of $\langle\rho_{3-i}\rangle$. We further select notation so that

$$I_i^1 \cap I_i^2 = \langle\rho_{3-i}\rangle.$$

The next lemma follows immediately from the 2-transitive action of F_i on the set $\{I_i^1, I_i^2, I_i^3, I_i^4\}$.

Lemma 7.5. *For $1 \le l \le 4$ and $1 \le j < k \le 4$ we have*

(i) *I_1^l has Type D- and $I_1^j \cap I_1^k \in \mathcal{M}(I_1)$; and*
(ii) *I_2^l has Type D+ and $I_2^j \cap I_2^k \in \mathcal{P}(I_2)$.*

□

With these comments we have the following lemma directly from Lemmas 6.3 and 6.4.

Lemma 7.6. *For $i = 1, 2$ and for $1 \le j < k \le 4$, we have*

$$C_G(I_i^k \cap I_i^j) \cong 3 \times \mathrm{Aut}(\mathrm{SU}_4(2)) \text{ or } 3 \times \mathrm{Sp}_6(2).$$

Furthermore, the isomorphism type of $C_G(I_i^k \cap I_i^j)$ does not depend on i, j or k.

Recall the Type N+ subgroups of order 9 are just the non-degenerate subgroups of J of plus type.

Lemma 7.7. *$I_1 \cap I_2$ is of Type N+.*

Proof. We know that $I_1 \cap I_2 = C_J(\langle r_1, r_2\rangle)$ and is consequently non-degenerate. Since $Z \le I_1 \cap I_2$, it has Type N+. $\square$

The next lemma is an adaptation of Lemma 5.3(ii) to K_i.

Lemma 7.8. *$F_i = N_{K_i}(I_i)$ controls K_i-fusion of elements in I_i.*

Proof. By Lemma 7.2, $C_S(r_i) \in \mathrm{Syl}_3(K_i)$ and thus I_i is the Thompson subgroup of $C_S(r_i)$ and is elementary abelian. It follows from [1, 37.6] that $N_{K_i}(I_i)$ controls fusion in I_i. As $C_G(I_i) \le M$, we calculate that $C_G(I_i) = J\langle r_i\rangle$. Hence $C_{K_i}(I_i) = I_i\langle r_i\rangle$ and $N_{K_i}(I_i) = L \cap K_i = F_i$. $\square$

For $i \in \{1, 2\}$ and $1 \le j < k \le 4$,

$$E_i^{j,k} = E(C_G(I_i^j \cap I_i^k)).$$

So $E_i^{j,k} \cong \mathrm{SU}_4(2)$ or $\mathrm{Sp}_6(2)$ and we note again that the isomorphism type of this group does not depend on i, j or k. At least one potential avenue for confusion is caused by this notation so please note that $E_i^{j,k}$ does not centralize r_i. Rather it centralizes a conjugate of r_{3-i}. Indeed $E_1^{1,2} = E(C_G(\rho_2))$ centralizes r_2 and $E_2^{1,2} = E(C_G(\rho_1))$ centralizes r_1 by Lemma 6.3.

Notice that I_i is centralized by r_i and so r_i is contained in $C_G(I_i^j \cap I_i^k)$ and it centralizes $I_i \cap E_i^{j,k}$ and this contains Z. It follows that $I_i \cap E_i^{j,k}$ is of Type N+ as it must also be non-degenerate. This means that r_i acts as an involution of type a_2 on $E_i^{j,k}$ in the sense of Table 1. Therefore, Lemma 2.2(ii) gives the following result:

Lemma 7.9. *We have*

$$\begin{aligned} C_{K_i}(I_i^j \cap I_i^k) &= C_{C_G(I_i^j \cap I_i^k)}(r_i) \\ &\approx \begin{cases} 3 \times 2_+^{1+4}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3)) & E_i^{j,k} \cong \mathrm{SU}_4(2) \\ 3 \times 2^{1+2+4}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3)) & E_i^{j,k} \cong \mathrm{Sp}_6(2) \end{cases}. \end{aligned}$$

$\square$

The next lemma now is the key. It shows that the groups $O_2(C_{K_i}(I_j^i))$ are quaternion groups of order eight which pairwise commute and so generate an extraspecial group of order 2^9.

Lemma 7.10. *Assume that $i = 1, 2$ and $1 \le j < k \le 4$.*

(i) *For $m \in \{j, k\}$, $I_i^m \cap E_i^{j,k}$ is a 3-central element of G and of $E_i^{j,k}$;*

(ii) $C_G(I_i^k) = (I_i^k \cap I_i^j) \times C_{E_i^{j,k}}(I_k \cap E_i^{j,k}) \approx 3 \times 3_+^{1+2}.\mathrm{SL}_2(3)$*;*

(iii) (a) $O_2(C_{K_i}(I_i^j)) \cong O_2(C_{K_i}(I_i^k)) \cong \mathrm{Q}_8$*;*
(b) $O_2(C_{K_i}(I_i^j))O_2(C_{K_i}(I_i^k)) \le O_2(C_{K_i}(I_i^j \cap I_i^k))$ *with equality if $E_i^{j,k} \cong \mathrm{SU}_4(2)$; and*
(c) $[O_2(C_{K_i}(I_i^j)), O_2(C_{K_i}(I_i^k))] = 1$*; and*

(iv) $C_{I_i}(O_2(C_{K_i}(I_i^j))O_2(C_{K_i}(I_i^k))) = I_i^j \cap I_i^k$.

Proof. It suffices to prove part (i) for I_i^1 as then the result will follow by conjugating by F_i

So consider $I_i^1 \cap I_i^2 = \langle \rho_{3-i} \rangle$. Then, by Lemma 6.2, $C_S(\rho_{3-i}) = Q_i J$ and $C_S(\rho_{3-i})' \cap Z(C_S(\rho_{3-i})) = Z$. Thus $Z \le I_i^1 \cap E_i^{1,j}$ is 3-central in G and in $E_i^{1,j}$. Part (i) follows as F_i acts 2-transitively on $\{I_i^j \mid 1 \le j \le 4\}$.

Part (ii) follows from (i) as the centralizer of a 3-central element in $\mathrm{Sp}_6(2)$ and $\mathrm{SU}_4(2)$ has shape $3_+^{1+2}.\mathrm{SL}_2(3)$.

To deduce part (iii), we first note that

$$O_2(C_{K_i}(I_i^k)) \cong O_2(C_{K_i}(I_i^j)) \cong \mathrm{Q}_8$$

follows from (ii) as r_i is an involution in $C_G(I_i^k)$. We have $l \in \{j, k\}$, $O_2(C_{K_i}(I_i^l)) \le C_{K_i}(I_i^j \cap I_i^k)$ and is normalized by $I_i^j I_i^k = I_i$. Since

$$\begin{aligned} C_{K_i}(I_i^j \cap I_i^k) &= C_{C_G(I_i^j \cap I_i^k)}(r_i) \\ &\approx \begin{cases} 3 \times 2_+^{1+4}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3)) & E_i^{j,k} \cong \mathrm{SU}_4(2) \\ 3 \times 2^{1+2+4}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3)) & E_i^{j,k} \cong \mathrm{Sp}_6(2) \end{cases} \end{aligned}$$

by Lemma 7.9, it follows that $O_2(C_{K_i}(I_i^l)) \le O_2(C_{K_i}(I_i^j \cap I_i^k))$. Now we apply Lemma 2.5(iii) to see that $[O_2(C_{K_i}(I_i^k)), O_2(C_{K_i}(I_i^k))] = 1$. (Recall that $O_2(C_{\mathrm{SU}_4(2)}(r_i)) \le O_2(C_{\mathrm{Sp}_6(2)}(r_i))$.)

Part (iv) follows as $I_i \cap E_i^{j,k}$ acts faithfully on $O_2(C_{K_i}(I_i^j))O_2(C_{K_i}(I_i^k))$. $\square$

We now introduce some further notation

Notation 7.11. *For $i = 1, 2$, $1 \le k \le 4$,*

$$\Sigma_i^k = O_2(C_{K_i}(I_i^k)) \cong \mathrm{Q}_8$$

and

$$\Sigma_i = \langle \Sigma_i^k \mid 1 \le k \le 4\rangle = \langle O_2(C_{K_i}(I_i^k)) \mid 1 \le k \le 4\rangle.$$

Note that $\Sigma_1^1 = O_2(C_{K_1}(A_2)) = R_1$ and $\Sigma_2^1 = O_2(C_{K_2}(A_1)) = R_2$.

Lemma 7.12. *We have Σ_i is extraspecial of order 2^9 and plus type, $Z(\Sigma_i) = \langle r_i\rangle$ and $F_i/\langle r_i\rangle$ acts faithfully on Σ_i.*

Proof. The structure of Σ_i follows from Lemma 7.10 (iii) as the generating subgroups commute pairwise. To see the last part is suffices to show that I_i acts faithfully on Σ_i as every normal subgroup of F_i which strictly contains $\langle r_i\rangle$ contains I_i. Using Lemma 7.10 (iv) we see that $C_{I_i}(\Sigma_i) = \bigcap_{j=1}^4 I_i^j = 1$. □

At this stage we have constructed the extraspecial group of order 2^9 on which F_i acts.

Lemma 7.13. *The following hold:*

(i) $C_{\Sigma_1}(Z) = R_1$, $C_{\Sigma_1}(I_1^j \cap I_1^k) = \Sigma_1^j\Sigma_1^k$ *and, if* $\langle x\rangle \in \mathcal{P}(I_1)$, *then* $C_{\Sigma_1}(x) = \langle r_1\rangle$.

(ii) $C_{\Sigma_2}(Z) = R_2$, $C_{\Sigma_2}(I_2^j \cap I_2^k) = \Sigma_2^j\Sigma_2^k$ *and, if* $\langle x\rangle \in \mathcal{M}(I_2)$, *then* $C_{\Sigma_2}(x) = \langle r_2\rangle$.

Proof. We prove (i) the proof of (ii) being the same. Let $1 \le j \le 4$. We know that $\Sigma_1 = \Sigma_1^1\Sigma_1^2\Sigma_1^3\Sigma_1^4$. Since I_1 acts faithfully on Σ_1, we have that $C_{I_1}(\Sigma_1^j) = I_1^j$. Thus the elements of $\mathcal{P}(I_1)$ act non-trivially on each Σ_1^j and so $C_{\Sigma_1}(x) = \langle r_1\rangle$ for $\langle x\rangle \in \mathcal{P}(I_1)$. Since we know that Z centralizes exactly $R_1 = \Sigma_1^1$ on Σ_1 we now have that (i) holds. □

8. The structure of $C_G(\rho_1)$

We continue to use our standard notation. In this section we are going to show that $C_G(\rho_1)$ is isomorphic to the corresponding centralizer in $\mathrm{F}_4(2)$. So our aim is to show that $C_G(\rho_1) \cong 3 \times \mathrm{Sp}_6(2)$. By Lemma 6.3 we have that $C_G(\rho_1)$ either is as in $\mathrm{F}_4(2)$ or is isomorphic to $3 \times \mathrm{Aut}(\mathrm{SU}_4(2))$. We will show the latter case yields a contradiction.

Lemma 8.1. *Suppose that $C_G(\rho_i) \cong 3 \times \mathrm{Aut}(\mathrm{SU}_4(2))$. Then Σ_i is the unique maximal signalizer for I_i^1 in K_i.*

Proof. We simplify our notation by assuming that $i = 1$. The argument for $i = 2$ is the same. Notice that

$$\{I_1^1 \cap I_1^j \mid 2 \le j \le 4\} = \mathcal{M}(I_i^1).$$

The only other proper subgroup of I_1^1 is Z by Lemma 7.5. Hence, as $E_1^{1,j} \cong \mathrm{SU}_4(2)$ by assumption, Lemma 7.10 (iii)(b) implies that

$$\Sigma_1 \ge O_2(C_{K_1}(I_1^k \cap I_1^j)) = O_{3'}(C_{K_1}(I_1^k \cap I_1^j)).$$

Suppose that Θ is a signalizer for I_1^1. Then
$$\Theta = \langle C_\Theta(a) \mid a \in I_1^{1\#}\rangle.$$
However,
$$C_\Theta(Z) \le O_{3'}(M \cap K_1) = R_1 \le \Sigma_1$$
and, for $1 < j \le 4$, by Lemma 7.9,
$$C_\Theta(I_1^1 \cap I_1^j) \le O_{3'}(C_{K_i}(I_1^1 \cap I_1^j)) = \Sigma_1\Sigma_j \le \Sigma_1.$$
Hence $\Theta \le \Sigma_1$. □

The next lemma puts us firmly on the track of $\mathrm{F}_4(2)$ and $\mathrm{Aut}(\mathrm{F}_4(2))$.

Lemma 8.2. *We have $C_G(\rho_1) \cong C_G(\rho_2) \cong 3 \times \mathrm{Sp}_6(2)$.*

Proof. Suppose that the lemma is false. Then by Lemmas 6.3 and 6.4
$$C_G(\rho_1) \cong C_G(\rho_2) \cong 3 \times \mathrm{Aut}(\mathrm{SU}_4(2)).$$
We claim that, for $i = 1, 2$, Σ_i is self-centralizing in K_i. Let $W_i = C_G(\Sigma_i)$. Then $W_i \le K_i$ and, as $C_S(r_i) \in \mathrm{Syl}_3(K_i)$ by Lemma 7.2 and since this group acts faithfully on Σ_i by Lemma 7.12, we have that W_i is a $3'$-group which is normalized by I_i^1. By Lemma 8.1, Σ_i is the unique maximal signalizer for I_i^1 and hence $\Sigma_i \ge W_i$.

Since Σ_i is the unique maximal signalizer for I_i^1 in K_i it is also the unique maximal signalizer of $Q_{3-i} \ge I_i^1$ and $I_i \ge I_i^1$ in K_i. It follows that $N_G(\Sigma_i) \ge \langle F_i, C_M(r_i)\rangle$ as Q_{3-i} is a normal subgroup of $C_M(r_i)$. Now
$$C_M(r_i)\Sigma_i/\Sigma_i = I_iQ_{3-i}R_{3-i}\langle f\rangle\Sigma_i/\Sigma_i$$
as $R_i \le \Sigma_i$. We now deduce $C_{C_M(Z)}(r_i)\Sigma_i/\Sigma_i$ is isomorphic to a 3-centralizer in $\mathrm{PSp}_4(3)$. Furthermore, as Σ_i is the unique maximal signalizer for I_i in K_i, we have that I_i does not normalize any non-trivial $3'$-subgroup of $N_G(\Sigma_i)/\Sigma_i$ and f inverts Z. Therefore, since $F_i \le N_G(\Sigma_i)$, Prince's Theorem 2.9 yields
$$N_G(\Sigma_i)/\Sigma_i \cong \mathrm{Aut}(\mathrm{SU}_4(2)) \text{ or } \mathrm{Sp}_6(2).$$
Observe that $N_G(\Sigma_i) \ge \langle F_i, C_M(r_i)\rangle \ge E(C_G(\rho_i))$.

We claim $N_G(\Sigma_i) = K_i$. To prove this we intend to apply Theorem 2.17 to $K_i/\langle r_i\rangle$. We have already verified hypotheses (i) and (ii) of that theorem.

As $N_G(\Sigma_i)/\Sigma_i \cong \mathrm{Aut}(\mathrm{SU}_4(2))$ or $\mathrm{Sp}_6(2)$, every element of $C_S(r_i)\Sigma_i/\Sigma_i$ is $N_G(\Sigma_i)/\Sigma_i$-conjugate to an element of $I_i\Sigma_i/\Sigma_i = J(C_S(r_i))\Sigma_i/\Sigma_i$ the Thompson subgroup of $C_S(r_i)\Sigma_i/\Sigma_i$. Since F_i controls fusion in I_i by Lemma 7.8, we also have hypothesis (iii) of Theorem 2.17.

Again to simplify notation, assume that $i = 1$. Suppose that d is an element of order 3 with $d \in N_G(\Sigma_1) \cap N_G(\Sigma_1)^h$ for some $h \in K_1$ such that $C_{\Sigma_1}(d) \neq \langle r_1 \rangle$. Then, by Lemma 7.13 (i), we may suppose that $\langle d \rangle = Z$ or $\langle d \rangle = I_1^1 \cap I_1^2 = \langle \rho_2 \rangle$. Then, as $N_{K_1}(Z) = C_M(r_1) \leq N_G(\Sigma_1)$ and $C_{K_1}(\rho_2) = C_{C_G(\rho_2)}(r_1) \leq N_G(\Sigma_1)$, we deduce

$$C_{K_1}(d) \leq N_G(\Sigma_1).$$

On the other hand, $C_{N_G(\Sigma_1)^h}(d)$ contains a K_1-conjugate X of I_1. Since $X \leq C_{K_1}(d) \leq N_G(\Sigma_1)$, we may suppose that $N_G(\Sigma_1) \cap N_G(\Sigma_1)^h \geq I_1$. But then $\Sigma_1 = \Sigma_1^h$ and $N_G(\Sigma_1) = N_G(\Sigma_1)^h$ as Σ_1 is the unique maximal signalizer for I_1 in K_1 by Lemma 8.1. Thus the hypothesis of Theorem 2.17 fulfilled and therefore $K_1 = N_G(\Sigma_1)$.

Suppose that $N_G(\Sigma_1)/\Sigma_1 \cong \mathrm{Aut}(\mathrm{SU}_4(2))$. Let $\widetilde{\rho}_1 \in \mathcal{P}(I_1)$. Then, as $|\mathcal{P}(I_1)| = 3$,

$$C_{N_G(\Sigma_1)/\Sigma_1}(\widetilde{\rho}_1\Sigma_1) \cong 3^3.\mathrm{Dih}(8)$$

by Lemma 5.7 (v). On the other hand, by Lemma 7.4 this group is non-soluble which is a contradiction. We conclude that $N_G(\Sigma_1)/\Sigma_1 \cong \mathrm{Sp}_6(2)$. Repeating the arguments for $N_G(\Sigma_2)$ yields $N_G(\Sigma_2)/\Sigma_2 \cong \mathrm{Sp}_6(2)$. Furthermore, the elements from $\mathcal{P}(I_1)$ act fixed point freely on $\Sigma_1/\langle r_1 \rangle$ and the elements of $\mathcal{M}(I_2)$ act fixed point freely on $\Sigma_2/\langle r_2 \rangle$. In both cases, $i = 1, 2$, $\Sigma_i/\langle r_i \rangle$ is the spin module for $N_G(\Sigma_i)/\Sigma_i$.

Since r_2 commutes with $I_1 \cap I_2 \leq N_G(\Sigma_1)$ which has Type N+ by Lemma 7.7, Table 1 indicates that r_2 acts as a unitary transvection on $\Sigma_1/\langle r_1 \rangle$. Therefore $|C_{\Sigma_1/\langle r_1 \rangle}(r_2)| = 2^6$ and

$$2^6 \leq |C_{\Sigma_1}(r_2)| \leq 2^7.$$

Since $\langle r_1, r_2 \rangle$ is centralized by $I_1 \cap I_2$, $C_{\Sigma_1}(r_2)$ is $(I_1 \cap I_2)$-invariant. Because the elements of $\mathcal{P}(I_1 \cap I_2)$ act fixed point freely on $\Sigma_1/\langle r_1 \rangle$ (see Lemma 2.4) we infer that $|C_{\Sigma_1}(r_2)| = 2^7$. Now, as $K_i = N_G(\Sigma_i)$ for $i = 1, 2$, $C_{\Sigma_1}(r_2)$ normalizes $C_{\Sigma_2}(r_1)$ and vice versa, and so

$$[C_{\Sigma_1}(r_2), C_{\Sigma_2}(r_1)] \leq \Sigma_1 \cap \Sigma_2.$$

Since $r_1 \notin \Sigma_2$ and $r_2 \notin \Sigma_1$, $\Sigma_1 \cap \Sigma_2$ is abelian and is centralized by $C_{\Sigma_1}(r_2)C_{\Sigma_2}(r_1)$. In particular, $\Sigma_1 \cap \Sigma_2 \leq Z(C_{\Sigma_1}(r_2))$. Thus, as $|C_{\Sigma_1}(r_2)| = 2^7$ and Σ_1 is extraspecial it follows that $\Sigma_1 \cap \Sigma_2$ has order at most 2^2 as $r_1 \notin \Sigma_2$. We have that $I_1 \cap I_2$ acts on $\Sigma_1 \cap \Sigma_2$. Since $|I_1 \cap I_2| = 3^2$, there is $w \in C_{I_1 \cap I_2}(\Sigma_1 \cap \Sigma_2)^\#$. Now $(\Sigma_1 \cap \Sigma_2)\langle r_1 \rangle$ is elementary abelian. Since, for $a \in \mathcal{S}(I_1 \cap I_2)$, we have $C_{\Sigma_1}(a) \cong \mathrm{Q}_8$ and, for $a \in \mathcal{P}(I_1 \cap I_2)$, we have $C_{\Sigma_1}(a) = \langle r_1 \rangle$, we must have $\langle w \rangle \in \mathcal{M}(I_1 \cap I_2)$. But then $\Sigma_1 \cap \Sigma_2 \leq C_{\Sigma_2}(w) = 1$ by Lemma 7.13. This means that $\Sigma_1 \cap \Sigma_2 = 1$ which then forces $[C_{\Sigma_1}(r_2), C_{\Sigma_2}(r_1)] = 1$ and Lemma 2.2 (iv) provides a contradiction. $\square$

9. Some subgroups in the centralizer of the involutions r_1 and r_2

In this section, we finally construct $O_2(K_i)$ where $K_i = C_G(r_i)$. Recall from Definition 3.1, we expect $O_2(K_i)$ to be a product of an elementary abelian group of order 2^7 by an extraspecial group of order 2^9. We have already located the extraspecial group Σ_i. In this section we uncover the elementary abelian group. We consider the situation for K_1. In the previous section we proved that $C_G(\rho_2) \cong 3 \times \mathrm{Sp}_6(2)$. With this additional information we study $C_{K_1}(\rho_2)$. This group has shape $3 \times 2^{1+2+4}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3))$. For us it is important that $Z(O_2(C_{K_1}(\rho_2)))$ is elementary abelian of order 8. Furthermore $I_1 = C_J(r_1)$ normalizes this group. This time there are six conjugates of this group under the action $C_L(r_1)$ and we define a group Υ_1 generated by these six conjugates. We show that Υ_1 is elementary abelian of order 2^7 and centralizes Σ_1, the extraspecial group found earlier. Hence the product of both gives a 2-group Γ_1 of order 2^{15}, which is in fact isomorphic to the corresponding group in $\mathrm{F}_4(2)$. Furthermore we show that $N_G(\Gamma_1)/\Gamma_1 \cong \mathrm{Sp}_6(2)$ and so $N_G(\Gamma_1)$ is similar to a 2-centralizer in $\mathrm{F}_4(2)$. In the next section show $K_1 = N_G(\Gamma_1)$.

We use our, by now, standard notation. In particular recall the definition of Σ_i from 7.11 and I_i^j the conjugates of A_{3-i} under $F_i = C_L(r_i)$. Our first goal is to construct a signalizer for I_i^1, $i = 1, 2$, which contains Σ_i properly. So, for $1 \le j < k \le 4$, we define

$$\Theta_i^{j,k} = Z(O_2(C_{K_i}(I_i^j \cap I_i^k)))$$

and put

$$\Upsilon_i = \langle \Theta_i^{j,k} \mid 1 \le j < k \le 4 \rangle.$$

We will shortly show that Υ_i is elementary abelian of order 2^7.

As $C_G(I_i^j \cap I_i^k) \cong 3 \times \mathrm{Sp}_6(2)$, Lemma 7.9 yields

$$C_{K_i}(I_i^j \cap I_i^k) \approx 2^{1+2+4}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3)).$$

Hence, by Lemmas 2.5 (iii) and (iv) and 7.10(iii), $\Theta_i^{j,k}$ is elementary abelian of order 2^3 and

$$O_2(C_{K_i}(I_i^j \cap I_i^k)) = \Sigma_i^j \Sigma_i^k \Theta_i^{j,k}.$$

We record this latter equality.

Lemma 9.1. *For $i = 1, 2$ and $1 \le j < k \le 4$, $O_2(C_{K_i}(I_i^j \cap I_i^k)) = \Sigma_i^j \Sigma_i^k \Theta_i^{j,k}$.* □

Lemma 9.2. *Suppose that $i = 1, 2$ and $\{j, k, l, m\} = \{1, 2, 3, 4\}$. Then*

(i) $\Theta_i^{j,k}$ *is elementary abelian of order* 2^3*, contains* r_i *and a* G*-conjugate* s_{3-i} *of* r_{3-i} *with* $s_{3-i} \neq r_i$.
(ii) $\Theta_i^{j,k} = \Theta_i^{l,m}$.
(iii) Υ_i *centralizes* Σ_i.
(iv) $\Theta_i^{j,k}\Theta_i^{k,l}$ *is elementary abelian of order* 2^5.
(v) Υ_i *is elementary abelian of order* 2^7 *and is normalized by* I_i.

Proof. To reduce the notational complexity of our argument we present the proof for $i = 1$ the proof when $i = 2$ is the same but we have to be careful when following the members of $\mathcal{M}(J)$ and $\mathcal{P}(J)$ in the arguments.

By definition
$$\Theta_1^{j,k} = Z(O_2(C_{K_1}(I_1^j \cap I_1^k))).$$
We know $I_1^j \cap I_1^k \in \mathcal{M}(J)$ from Lemma 7.5 and we know $C_{K_1}(I_1^j \cap I_1^k) \cap E_1^{j,k}$ is a line stabiliser in the natural symplectic representation of $E_1^{j,k} \cong \mathrm{Sp}_6(2)$. Thus $\Theta_1^{j,k}$ is elementary abelian of order 2^3 by Lemma 2.5 and of course $\Theta_1^{j,k}$ contains r_1 and, by Lemma 7.4, r_2 is a 2-central involution in E_1^{jk} and so $\Theta_1^{j,k}$ also contains a conjugate of r_2. This proves (i).

Now $J \cap E_1^{j,k}$ centralizes a conjugate of r_2 and is thus G-conjugate to I_2. It follows from Lemma 5.7 that $|\mathcal{S}(J \cap E_1^{j,k})| = 4$, $|\mathcal{P}(J \cap E_1^{j,k})| = 6$ and $|\mathcal{M}(J \cap E_1^{j,k})| = 3$. Now using Lemma 2.5 (iv), we have
$$X_1^{j,k} = C_{I_1 \cap E_1^{j,k}}(\Theta_1^{j,k}) \in \mathcal{M}(I_1 \cap E_1^{j,k}).$$
Observe $X_1^{j,k} \leq I_1$ and so $X_1^{j,k}$ normalizes Σ_1.

Since $X_1^{j,k} \in \mathcal{M}(I_1)$, $C_{\Sigma_1}(X_1^{j,k})$ has order 2^5 by Lemma 7.13. As $[\Sigma_1^j\Sigma_1^k, X_1^{j,k}] = \Sigma_1^j\Sigma_1^k$ and Σ_1 is extraspecial, we deduce
$$C_{\Sigma_1}(X_1^{j,k}) = \Sigma_1^l\Sigma_1^m = C_{\Sigma_1}(\Sigma_1^j\Sigma_1^k).$$
In particular, we now have $X_1^{j,k} = I_1^l \cap I_1^m$ by Lemma 7.13. This implies $\Theta_1^{j,k} \leq C_G(I_1^l \cap I_1^m)$ and $\Theta_1^{j,k}$ is normalized by I_1; therefore
$$\langle \Theta_1^{j,k}, \Sigma_1^l\Sigma_1^m \rangle = O_2(C_{K_1}(I_1^l \cap I_1^m)).$$
Since $\Theta_1^{j,k}$ is I_1-invariant and elementary abelian, we infer $\Theta_1^{j,k} = \Theta_1^{l,m}$ and that $\Theta_1^{j,k}$ commutes with $\Sigma_1^j\Sigma_1^k$ as well as with $\Sigma_1^l\Sigma_1^m$. Since $\Sigma_1 = \Sigma_1^j\Sigma_1^k\Sigma_1^l\Sigma_1^m$, we have now proved claims (ii) and (iii).

Because $\Theta_1^{j,k} = \Theta_1^{l,m}$ we have that $\Theta_1^{j,k}$ is centralized by $\langle X_1^{j,k}, X_1^{l,m} \rangle = \langle I_1^i \cap I_1^j, I_1^l \cap I_1^m \rangle$ which has Type N- as $\Theta_1^{j,k}$ does not commute with a conjugate of Z. Hence $\langle \Theta_1^{j,k}, \Theta_1^{k,l} \rangle$ is centralized by
$$Y = \langle X_1^{j,k}, X_1^{l,m} \rangle \cap \langle X_1^{k,l}, X_1^{j,m} \rangle \in \mathcal{P}(J).$$

Now $C_G(Y) \cong 3 \times \mathrm{Sp}_6(2)$ and $I_1 \cap E(C_G(Y))$ is of Type N- by Lemma 7.4. Since $\langle \Theta_1^{j,k}, \Theta_1^{k,l}\rangle$ centralizes r_1 and is normalized by I_1 we infer that r_1 is an involution of $E(C_G(Y))$ with centralizer of shape $2^5.\mathrm{Sp}_4(2)$ and that $\langle \Theta_1^{j,k}, \Theta_1^{k,l}\rangle \le O_2(C_{E(C_G(Y))}(r_1))$ which is elementary abelian. But then

$$\langle \Theta_1^{j,k}, \Theta_1^{k,l}\rangle = \Theta_1^{j,k}\Theta_1^{k,l}$$

is elementary abelian of order at most 2^5. It now follows that $\Upsilon_1 = \Theta_1^{1,2}\Theta_1^{2,3}\Theta_1^{2,4}$ has order at most 2^7 and is I_1-invariant. We have seen that $C_{I_1}(\Theta_1^{j,k}\Theta_1^{k,l})$ is $I_1^j \cap I_1^k$. Thus $C_{I_1}(\Upsilon_i) \le I_1^1 \cap I_1^2 \cap I_1^3 \cap I_1^4 = 1$. Hence I_1 acts faithfully on Υ_1 and so $|\Upsilon_1| = 2^7$. This completes the proof of (iv) and (v) and the verification of the statements in the lemma. $\square$

For $i = 1, 2$, we now set

$$\Gamma_i = \Sigma_i \Upsilon_i.$$

Lemma 9.3. *For $i = 1, 2$, we have that Γ_i has order 2^{15} and is normalized by F_i. Furthermore the following hold.*

(i) *$Z(\Gamma_i) = \Upsilon_i$; and*
(ii) *$[\Gamma_i, \Gamma_i] = \langle r_i\rangle$.*

Proof. By Lemmas 7.12 and 9.2, Σ_i has order 2^9 and is extraspecial and $|\Upsilon_i| = 2^7$ and centralizes Σ_i. This yields $\Upsilon_i \cap \Sigma_i = \langle r_i\rangle$ and Γ_i has order 2^{15}. Furthermore, as Σ_i is extraspecial, Υ_i is elementary abelian and Υ_i commutes with Σ_i we have that $\Upsilon_i = Z(\Gamma_i)$ and $[\Gamma_i, \Gamma_i] = \langle r_i\rangle$. Hence points (i) and (ii) hold.

By the construction of Σ_i and Υ_i, F_i normalizes both groups and consequently also normalizes their product, completing the proof. $\square$

Lemma 9.4. *For $i = 1, 2$, Γ_i is the unique maximal signalizer for I_i^1 in K_i.*

Proof. Assume that W is an I_i^1 signalizer in K_i. Then

$$W = \langle C_W(x) \mid x \in (I_i^1)^{\#}\rangle.$$

If $\langle x\rangle = Z \in \mathcal{S}(I_i^1)$, then

$$O_{3'}(C_{K_i}(Z)) = R_i = \Sigma_i^1 \le \Sigma_i \le \Gamma_i$$

is the unique maximal I_i^1 signalizer in $C_{K_i}(Z)$. All the other subgroups of order 3 in I_i^1 are conjugate to $\langle \rho_{3-i}\rangle$ by an element of $Q_{3-i} \le F_i$. Hence we only need to consider I_i^1 signalizers in $C_{K_i}(\rho_{3-i})$.

By Lemma 8.2, $C_G(\rho_{3-i}) = C_G(I_i^1 \cap I_i^2) \cong 3 \times \mathrm{Sp}_6(2)$ and we know from Lemma 7.9 that

$$C_{K_i}(\rho_{3-i}) \approx 3 \times 2^{1+2+4}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3)).$$

Set $D = C_{K_i}(\rho_{3-i})$. Then

$$O_2(D) = \Sigma_i^1\Sigma_i^2\Theta_i^{1,2} \leq \Gamma_i$$

and, Lemma 2.5(ii), implies $ZO_2(D)/O_2(D)$ is diagonal in $D/O_2(D)$. Since $C_W(\rho_{3-i})$ is normalized by Z we infer that $C_W(\rho_{3-i}) \leq \Gamma_i$ as claimed. □

Lemma 9.5. *For $i = 1, 2$, there is a G-conjugate of r_i in $\Gamma_i \setminus \Upsilon_i$.*

Proof. This fusion can already be seen in

$$C_{K_i}(\rho_{3-i}) \approx 3 \times 2^{1+2+4}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3))$$

as r_i is not weakly closed in $O_2(C_{K_i}(\rho_{3-i}))$ with respect to $C_G(\rho_{3-i})$ by Lemma 2.5 (vi). □

We are now able to determine the structure of $N_G(\Gamma_i)$.

Lemma 9.6. *For $i = 1, 2$, the following hold.*

(i) $N_G(\Gamma_i)/\Gamma_i \cong \mathrm{Sp}_6(2)$;
(ii) *as $N_G(\Gamma_i)/\Gamma_i$-modules, Γ_i/Υ_i is a spin module and $\Upsilon_i/\langle r_i\rangle$ is a natural module;*
(iii) $\mathrm{Syl}_2(N_G(\Gamma_i)) \subseteq \mathrm{Syl}_2(K_i)$; *and*
(iv) *if $T \in \mathrm{Syl}_2(N_G(\Gamma_i))$, then $\Gamma_i/\langle r_i\rangle = J(T/\langle r_i\rangle)$, $Z(T) \leq \Upsilon_i$ and $Z(T)$ has order 4.*

In particular, $N_G(\Gamma_i)$ is similar to a 2-centralizer in $\mathrm{F}_4(2)$.

Proof. We already know that Γ_i is normalized by F_i and we have that Γ_i is the unique maximal I_i^1-signalizer in K_i by Lemma 9.4. It follows that Γ_i is also the unique maximal signalizer for $Q_{3-i} \geq I_i^1$ in K_i. Therefore $N_{E(C_G(\rho_i))}(Q_{3-i})$ also normalizes Γ_i. It follows from [4, page 46] that

$$X = \langle F_i, N_{E(C_G(\rho_i))}(Q_{3-i})\rangle \cong \mathrm{Aut}(\mathrm{SU}_4(2))$$

and X normalizes Γ_i.

Since $C_{K_i}(Z)\Gamma_i/\Gamma_i$ is a 3-centralizer of type $\mathrm{PSp}_4(3)$, Γ_i is a maximal signalizer for I_i^1 and Z is inverted in $N_G(\Gamma_i)/\Gamma_i$, we deduce $N_G(\Gamma_i)/\Gamma_i \cong \mathrm{Sp}_6(2)$ or $\mathrm{Aut}(\mathrm{SU}_4(2))$ by using Theorem 2.9.

We know that I_i acts faithfully on both Γ_i/Υ_i and $\Upsilon_i/\langle r_i\rangle$. In particular, as $|\Upsilon_i/\langle r_i\rangle| = 2^6$, if $N_G(\Gamma_i)/\Gamma_i \cong \mathrm{Aut}(\mathrm{SU}_4(2))$ then $\Upsilon_i/\langle r_i\rangle$ is an orthogonal module and if $N_G(\Gamma_i)/\Gamma_i \cong \mathrm{Sp}_6(2)$ then $\Upsilon_i/\langle r_i\rangle$ is a natural module. Similarly since $C_{\Sigma_i}(Z) = \Sigma_i^1$ and since this subgroup is not normalized by F_i and $|\Gamma_i/\Upsilon_i| = 2^8$, if $N_G(\Gamma_i)/\Gamma_i \cong \mathrm{Aut}(\mathrm{SU}_4(2))$, then Γ_i/Υ_i is an natural module and, if $N_G(\Gamma_i)/\Gamma_i \cong \mathrm{Sp}_6(2)$, then Γ_i/Υ_i is a spin module (see Lemma 2.1). So once we have proved part (i), part (ii) will also be proved.

Next we prove (iii) and the first part of (iv). Let $T \in \mathrm{Syl}_2(N_G(\Gamma_i))$. Since, by Lemma 2.7, $\Gamma_i/\langle r_i\rangle$ is not an F-module for $N_G(\Gamma_i)/\Gamma_i$, [8, Lemma 26.15] implies that $\Gamma_i/\langle r_i\rangle$ is the Thompson subgroup of $T/\langle r_i\rangle$. It follows that $N_{K_i}(T) \le N_G(\Gamma_i)$ and, in particular, $T \in \mathrm{Syl}_2(K_i)$ and $N_{K_i}(T) = T$. Notice furthermore that $N_G(\Gamma_i)/\langle r_i\rangle$ controls $K_i/\langle r_i\rangle$-fusion in $\Gamma_i/\langle r_i\rangle$. The last two parts of (iv) follow from the fact that Σ_i is extraspecial and Lemma 2.8.

It remains to prove (i). Assume that $N_G(\Gamma_i)/\Gamma_i \cong \mathrm{Aut}(\mathrm{SU}_4(2))$. Using Lemma 9.5, there exists $g \in G$ and $s \in \Gamma_i \setminus \Upsilon_i$ such that $s = r_i^g$. Since $N_G(\Gamma_i^g)$ contains a Sylow 2-subgroup of $C_G(s)$, there is a $h \in C_G(s)$ such that $C_{\Gamma_1}(s)^h \le N_G(\Gamma_i^g)$ and we have $s = r_i^{gh}$ so we may suppose g was chosen so $C_{\Gamma_1}(s) \le N_G(\Gamma_i^g)$. Note that, as $s \in \Gamma_i \setminus \Upsilon_i$, s is conjugate in Γ_i to sr_i and, as $N_G(\Gamma_i)/\langle r_i\rangle$ controls $K_i/\langle r_i\rangle$-fusion in $\Gamma_i/\langle r_i\rangle$, s is not K_i-conjugate to an element of Υ_i.

Since $C_{\Gamma_1}(s)$ contains an extraspecial group of order 2^7 with derived group $\langle r_i\rangle$, and $\mathrm{Aut}(\mathrm{SU}_4(2))$ does not (by Lemma 2.2), we have $r_i \in \Gamma_i^g$. It follows that $C_{\Gamma_i^g}(r_i)$, which has index at most 2 in Γ_i^g, also contains an extraspecial group of order 2^7. As $T \in \mathrm{Syl}_2(K_i)$, there is $f \in K_i$ such that $C_{\Gamma_i^g}(r_i)^f = C_{\Gamma_i^{gf}}(r_i) \le T$. It follows that $s^f \in \Gamma_i \setminus \Upsilon_i$ and we may as well suppose that $s = s^f$ (though we may no longer have $C_{\Gamma_1}(s) \le N_G(\Gamma_i^g)$). With this choice of s, $|\Gamma_i^g : \Gamma_i^g \cap N_G(\Gamma_i)| \le 2$. Now

$$\Phi(\Gamma_i^g \cap \Gamma_i) \le \Phi(\Gamma_i^g) \cap \Phi(\Gamma_i) = \langle s\rangle \cap \langle r_i\rangle = 1$$

which means $\Gamma_i^g \cap \Gamma_i$ is elementary abelian. As Γ_i contains Σ_i which is extraspecial of order 2^9, this yields $|\Gamma_i^g \cap \Gamma_i| \le 2^{11}$ and so

$$|(\Gamma_i^g \cap N_G(\Gamma_i))\Gamma_i/\Gamma_i| \ge 2^3.$$

Further

$$[\Upsilon_i \cap \Gamma_i^g, N_G(\Gamma_i) \cap \Gamma_i^g] \le [\Gamma_i^g, \Gamma_i^g] \cap \Upsilon_i = \langle s\rangle \cap \Upsilon_i = 1.$$

Hence, as $|(\Gamma_i^g \cap N_G(\Gamma_i))\Gamma_i/\Gamma_i| \ge 2^3$, Lemma 2.2(iii) (which says that $\mathrm{Aut}(\mathrm{SU}_4(2))$ contains no fours group of unitary transvections) implies $|\Upsilon_1 \cap \Gamma_i^g| \le 2^5$. Therefore $|\Gamma_i \cap \Gamma_i^g| \le 2^9$. We have now shown $|(\Gamma_i^g \cap N_G(\Gamma_i))\Gamma_i/\Gamma_i| \ge 2^5$ which, as this group is elementary abelian and the 2-rank of $\mathrm{Aut}(\mathrm{SU}_4(2))$ is 4, is a contradiction. Therefore $N_G(\Gamma_i)/\Gamma_i \cong \mathrm{Sp}_6(2)$ and this completes the proof of part (i) and thereby also (ii). □

10. The centralisers of r_1 and r_2

In this section we finally determine the structure of $K_i = C_G(r_i)$. We will prove $K_i = N_G(\Gamma_i)$ and hence conclude that K_i is similar to a 2-centralizer in $\mathrm{F}_4(2)$. The plan is to show Υ_i is strongly closed in a Sylow 2-subgroup of K_i with respect to K_i and then to quote Goldschmidt's Theorem in the form of Lemma 2.19 to show that $K_i = N_G(\Gamma_i)$. To achieve this we study K_i-fusion of involutions. As most of the centralizers of involutions in $N_G(\Gamma_i)$ have order divisible by three, this will be reduced to fusion of 3-elements. Hence the first lemma we prove in this section will be that $N_G(\Gamma_i)$ is strongly 3-embedded in K_i, which means that we have control of fusion of elements of order 3 in K_i.

We use all our previous notation and furthermore for this section we set $H_i = N_G(\Gamma_i)$.

Lemma 10.1. *For $i = 1, 2$, H_i is strongly 3-embedded in K_i. In particular, H_i controls fusion of elements of order 3 in H_i.*

Proof. Suppose that $x \in H_i$ has order 3. We will show $C_{K_i}(x)$ normalizes Γ_i. Recall $C_S(r_i) \in \mathrm{Syl}_3(K_i)$ and $C_S(r_i) \leq F_i \leq H_i$ so $C_S(r_i)$ normalizes Γ_i. Since every element of order 3 in $C_S(r_i)$ is H_i-conjugate into I_i, we may suppose $x \in I_i$.

Again to simplify our notation slightly we consider the case when $i = 1$. Thus $|\mathcal{S}(I_1)| = 4$, $|\mathcal{M}(I_1)| = 6$ and $|\mathcal{P}(I_1)| = 3$ by Lemma 5.6. If $\langle x\rangle \in \mathcal{S}(I_1)$, then we may suppose that $\langle x\rangle = Z$. In this case, by Lemma 7.2

$$C_{K_1}(Z) = Q_2R_1R_2I_1 \leq H_1.$$

So suppose that $\langle x\rangle = \langle \rho_2\rangle \in \mathcal{M}(I_1)$. Then, by Lemma 9.1,

$$C_{K_1}(\rho_2) = \Sigma_1^1\Sigma_1^2\Theta_1^{1,2}N_{F_1}(I_1 \cap E_1^{12}) \leq \Gamma_1F_1 \leq H_1.$$

Suppose $\langle x\rangle = \widetilde{\rho_1} \in \mathcal{P}(I_1)$. Then, by Lemma 7.4, $C_{K_1}(\widetilde{\rho_1}) \approx 3 \times 2^5{:}\mathrm{Sp}_4(2)$ and this has the same order as $C_{H_1}(\widetilde{\rho_1})$. Thus $C_{K_1}(\widetilde{\rho_1}) \leq H_1$. Finally, $N_{K_1}(C_S(r_1)) \leq N_{K_1}(Z)$ and so H_1 is strongly 3-embedded in K_1 by [8, Lemma 17.11]. □

We next show $H_i = K_i$ for $i = 1, 2$. The proof is accomplished through a series of lemmas. It suffices to prove this with $i = 1$ as the proof for $i = 2$ is the same. By Lemma 9.6 (ii), $Z(H_1) = \langle r_1\rangle$, $\Upsilon_1/Z(H_1)$ is the natural $\mathrm{Sp}_6(2)$-module and Γ_1/Υ_1 is the spin module for $\mathrm{Sp}_6(2)$. Let T be a Sylow 2-subgroup of H_1. From Lemma 9.6 (iv) we have $T \in \mathrm{Syl}_2(K_1)$.

Lemma 10.2. (i) *If $x \in \Upsilon_1^\#$ and $s \in x^{K_1}$, then s and sr_1 are not K_1-conjugate.*
(ii) *Υ_1 is strongly closed in Γ_1 with respect to K_1.*

Proof. (i) Obviously, if $x = r_1$, the result is true. So we may suppose $x \in \Upsilon_1 \setminus \langle r_1\rangle$. Since H_1 acts transitively on $(\Upsilon_1/\langle r_1\rangle)^\#$, we may additionally assume $x\langle r_1\rangle \in C_{\Upsilon_1/\langle r_1\rangle}(T)$ which has order 2 by Lemma 2.3. As by Lemma 2.8 the preimage of $C_{\Upsilon_1/\langle r_1\rangle}(T)$ is centralized by T we have $x \in Z(T)$.

Suppose that x is K_1-conjugate to xr_1. Then as x and $xr_1 \in Z(T)$, this conjugation must happen in $N_{K_1}(T)$. Since $T \in \mathrm{Syl}_2(K_1)$, this is impossible and it follows that x is not K_1-conjugate to xr_1. This proves (i)

Now consider $y \in \Gamma_1 \setminus \Upsilon_1$. Then $[y, \Gamma_1] = \langle r_1\rangle$ and so y is conjugate to r_1y in Γ_1. Therefore (i) implies (ii). □

Lemma 10.3. *Let $x \in \Upsilon_1$, $g \in K_1$ and assume that $s = x^g$ with $s \in T \setminus \Gamma_1$. Then s normalizes an H_1-conjugate of $I_1\Gamma_1$ and Σ_1.*

Proof. Since in $H_1/\Gamma_1 \cong \mathrm{Sp}_6(2)$ every involution is conjugate into $N_{H_1/\Gamma_1}(I_1\Gamma_1/\Gamma_1)$, we may as well suppose that s normalizes $I_1\Gamma_1$. In particular by Lemma 7.12 we may additionally assume $\Sigma_1^s = \Sigma_1$. □

Lemma 10.4. *Let $x \in \Upsilon_1$, $g \in K_1$ and assume that $s = x^g$ with $s \in T \setminus \Gamma_1$. Then the following hold:*

(i) $C_{\Gamma_1/\Upsilon_1}(s) = C_{\Gamma_1}(s)\Upsilon_1/\Upsilon_1$; *and*

(ii) $C_{H_1}(s)$ *is a $3'$-group.*

Proof. By Lemma 10.3 we may assume that s normalizes both $I_1\Gamma_1$ and Σ_1. Let $w\Upsilon_1 \in C_{\Gamma_1/\Upsilon_1}(s)$ and write $w = w_*u$ where $w_* \in \Sigma_1$ and $u \in \Upsilon_1$. Then

$$[w, s] = [w_*u, s] = [w_*, s][u, s] \in \Upsilon_1.$$

As s normalizes Σ_1, this means that $[w_*, s] \in \Sigma_1 \cap \Upsilon_1 = \langle r_1\rangle$. Since x is not K_1-conjugate to sr_1, we deduce that w_* is centralized by s and this proves (i).

Suppose that $W \in \mathrm{Syl}_3(C_{H_1}(s))$ and let $U \in \mathrm{Syl}_3(C_{H_1}(x))$. Then, as $\Upsilon_1/\langle r_1\rangle$ is the natural $\mathrm{Sp}_6(2)$-module, U has order 3^2 by Lemma 2.3. Since by Lemma 10.1 H_1 is strongly 3-embedded in K_1 we know that $U \in \mathrm{Syl}_3(C_{K_1}(x))$ and so $U^g \in \mathrm{Syl}_3(C_{K_1}(s))$. Thus there exists $h \in C_{K_1}(s)$ so that $U^{gh} \geq W$. Consequently $W \leq H_1 \cap H_1^{gh}$. If $W \neq 1$, Lemma 10.1 yields $gh \in H_1$ which contradicts the fact that $s = x^{gh}$, $s \in T \setminus \Sigma_1\Upsilon_1$ and $x \in \Upsilon_1$. Hence $W = 1$, proving (ii). □

Suppose that $s^* \in s\Gamma_1$ is an involution which is conjugate to s in K_1.

Then $ws = s^*$ with $w \in \Gamma_1$. We claim that $w \in C_{\Gamma_1}(s)$. To see this we note that the other possibility is that $w^s = w^{-1} = wr_1$ and then we calculate

$$s^{*s} = (ws)^s = w^s s = w^{-1}s = wr_1s = s^*r_1$$

which contradicts Lemma 10.2(i).

Let $q \in C_{\Gamma_1}(s)$ and assume that $[w, q] \neq 1$. Then, by Lemma 9.3, $[w, q] = r_1$ and

$$s^{*q} = (ws)^q = w^q s = w[w, q]s = wsr_1 = s^* r_1,$$

which is also impossible. Therefore $w \in Z(C_{\Gamma_1}(s))$. Since s normalizes Σ_1 and Σ_1 is extraspecial, the Three Subgroup Lemma implies $Z(C_{\Sigma_1}(s)) = [\Sigma_1, s]$. Thus Lemma 10.2(i) implies that

Lemma 10.5. *Let $x \in \Upsilon_1$, $g \in K_1$ and assume that $s = x^g$ with $s \in T \setminus \Gamma_1$. If s is H_1-conjugate to $s^* = ws$ where $w \in \Gamma_1$, then $w \in Z(C_{\Gamma_1}(s)) \leq [\Gamma_1, s]\Upsilon_1$. In particular, $s\Upsilon_1$ is Γ_1/Υ_1-conjugate to $s^*\Upsilon_1$ and $C_{H_1/\Gamma_1}(s\Upsilon_1) = C_{H_1/\Upsilon_1}(s)\Gamma_1/\Gamma_1$.*

Now we are going to identify the involution $s\Gamma_1$ in $H_1/\Gamma_1 \cong \mathrm{Sp}_6(2)$.

Lemma 10.6. *Let $x \in \Upsilon_1$, $g \in K_1$ and assume that $s = x^g$ with $s \in T \setminus \Gamma_1$. Then $s\Gamma_1$ is an involution of type c_2 and all K_1-conjugates of x in $H_1 \setminus \Gamma_1$ project to elements of this type.*

Proof. By Lemma 2.2 (i), $s\Gamma_1$ is an involution of type a_2, b_1, b_3 or c_2 in $H_1/\Gamma_1 \cong \mathrm{Sp}_6(2)$. If $s\Gamma_1$ is of type b_3, then Lemma 2.2 implies that $[\Gamma_1/\langle r_1\rangle, s] = C_{\Gamma_1/\langle r_1\rangle}(s)$ and consequently 3 divides $|C_{H_1}(s)|$. Hence $s\Gamma_1$ is not of type b_3 by Lemma 10.4 (ii).

If $s\Gamma_1$ is of type b_1 or a_2, then, by Lemma 10.5, $|C_{H_1/\Upsilon_1}(s)|$ is divisible by 3^2. If $s\Gamma_1$ is of type a_2, then Lemma 2.2 implies

$$|C_{\Upsilon/\langle r_1\rangle}(s)/[\Upsilon/\langle r_1\rangle, s]| = 4$$

and so s is centralized by an element of order 3 contrary to Lemma 10.4 (ii). Thus $s\Gamma_1$ is not of type a_2. If $s\Gamma_1$ is of type b_1, then Lemma 2.2 yields $C_{\Upsilon/\langle r_1\rangle}(s)/[\Upsilon/\langle r_1\rangle, s]$ is the natural $\mathrm{Sp}_4(2)$-module and, as $\mathrm{Sp}_4(2)$ acts transitively on the non-trivial elements of this module, we again see s is centralized by a 3-element, a contradiction. Thus $s\Gamma_1$ must be of type c_2. $\square$

Lemma 10.7. Υ_1 *is strongly 2-closed in T with respect to K_1.*

Proof. Let $x \in \Upsilon_1$, $g \in K_1$ and assume that $s = x^g$ with $s \in T \setminus \Gamma_1$. By Lemma 10.6, s acts as an element of type c_2 on the natural $\mathrm{Sp}_6(2)$-module.

Let $F = C_{\Sigma_1}(s) = [\Sigma_1, s]$. Then F has order 2^5 by Lemma 2.2. Thus the coset Fs consists solely of conjugates of s and of sr_1 and $F \cap \Upsilon_1 = \langle r_1\rangle$.

Recall that we may suppose that $x \in Z(T)$. So s is a 2-central element of K_1. Hence, as F is a 2-group which centralizes s, F is contained in a Sylow 2-subgroup T_0 of K_1 which centralizes s. Let Γ_1^* be the preimage of $J(T_0/\langle r_1\rangle)$, $\Upsilon_1^* = Z(\Gamma_1^*)$ and $H^* = N_G(\Gamma_1^*)$. By Lemma 9.6 we have that Γ_1^* is conjugate to Γ_1 in K_1. Then also H^* is K_1-conjugate to H_1 and $H^*/\Gamma_1^* \cong \mathrm{Sp}_6(2)$.

Assume that $y \in F \setminus \langle r_1 \rangle$. Then ys is conjugate to either s or sr_1. In particular any coset of $\langle r_1 \rangle$ in F contains some y such that ys is conjugate to s in K_1. If $y \in \Gamma_1^*$, then, as $y \in \Gamma_1 \setminus \Upsilon_1$, Lemma 10.2 (ii) yields $y \notin \Upsilon_1^*$ and consequently we also have $ys \in \Gamma_1^* \setminus \Upsilon_1^*$ which contradicts Lemma 10.2. Thus $y \notin \Gamma_1^*$ and the coset $y\Gamma_1^*$ contains ys. We deduce with Lemma 10.6 that $y\Gamma_1^*$ is of type c_2 in $N_{K_1}(\Gamma_1^*)/\Gamma_1^*$ and $F\Gamma_1^*/\Gamma_1^*$ is a subgroup of order 2^4 in which all the non-trivial elements are in class c_2. Since $\mathrm{Sp}_6(2)$ has no such subgroups by Lemma 2.2, we have a contradiction. Therefore Υ_1 is strongly 2-closed in T with respect to K_1. □

Next we can prove the main result of this section:

Lemma 10.8. *For $i = 1, 2$, we have $H_i = K_i$. In particular, K_1 and K_2 are similar to 2-centralizers in* $\mathrm{F}_4(2)$.

Proof. Again it is enough to prove the lemma for $i = 1$. By Lemma 10.7 we have that Υ_1 is strongly 2-closed in T with respect to K_1. Therefore Lemma 2.19 yields $K_1 \le N_G(\Upsilon_1)$. Now $C_{K_1}(\Upsilon_1) \cap C_S(r_1) = 1$ and so $C_{K_1}(\Upsilon_1)$ is a $3'$-group. Since, by Lemma 9.4, Γ_1 is the unique maximal I_1^1-signalizer in K_1, we conclude $\Gamma_1 \ge C_{K_1}(\Upsilon_1)$ and thus $\Gamma_1 = C_{K_1}(\Upsilon_1)$. It follows that $K_1 = N_{K_1}(\Upsilon_1) = N_{K_1}(\Gamma_1)$ as claimed. □

11. PROOF OF THEOREM 1.2

Having determined the shapes of the centralizers of the involutions r_1 and r_2, in this section we accomplish the final identification of G.

Let $T \in \mathrm{Syl}_2(K_1)$, where $K_1 = C_G(r_1)$, and recall that $\Gamma_1 = \Sigma_1\Upsilon_1 = O_2(K_1)$. The conclusion of the work of the previous sections is that K_1 is similar to a 2-centralizer in $\mathrm{F}_4(2)$.

By Lemma 9.2, Υ_1 contains a G-conjugate s_2 of r_2 with $s_2 \neq r_1$. As K_1 acts transitively on the non-trivial elements of $\Upsilon_1/\langle r_1 \rangle$, Lemma 2.8 shows that we may further suppose that $s_2 \in Z(T)$ and $Z(T) = \langle r_1, s_2 \rangle$. Define $U_2 = C_G(s_2)$. We have U_2 is G-conjugate to $K_2 = C_G(r_2)$ and thus, as $|K_1| = |K_2|$, we have $T \in \mathrm{Syl}_2(U_2)$.

We will use the two groups to construct a subgroup $P = \langle K_1, U_2 \rangle \cong F_4(2)$ using Theorem 3.3. Recall Definition 3.2, and note that K_1, U_2, T is an F_4 set-up.

Lemma 11.1. $P = \langle K_1, U_2 \rangle \cong \mathrm{F}_4(2)$.

Proof. This follows directly from Theorem 3.3. □

In fact we have the following corollary:

Corollary 11.2. *If X is any group which satisfies the assumptions of Theorem 1.2, then X contains a subgroup isomorphic to* $\mathrm{F}_4(2)$.

Proof. This follows immediately from Lemma 11.1. $\square$

Our aim is to show that G is isomorphic to either $\mathrm{F}_4(2)$ or $\mathrm{Aut}(\mathrm{F}_4(2))$. For this we will show that P is normal in G. As a first step we show that P is normalized by M and that $P_0 = PM$ is either $\mathrm{F}_4(2)$ or $\mathrm{Aut}(\mathrm{F}_4(2))$. We then produce a normal subgroup G_* of G of index at most two such that $P_0 \cap G_* = P$. Our objective is then to show $G_* = P$. This will be done using Holt's Theorem (Lemma 2.20). Hence we have to gain control of G_*-fusion of involutions in P. For this we show that P_0 is strongly 3-embedded in G_*, which will imply that P controls G_*-fusion in P. We start with the proof that M normalizes P.

We have $C_P(\rho_1) \cong C_P(\rho_2) \cong 3 \times \mathrm{Sp}_6(2)$ and so, by Lemma 8.2, $C_G(\rho_i) = C_P(\rho_i)$, $i = 1, 2$. As $\langle C_M(\rho_1), C_M(\rho_2)\rangle = M \cap P$, we see $\langle C_G(\rho_1), C_G(\rho_2)\rangle$ satisfies the assumptions of Theorem 1.2. By Corollary 11.2 we get that $\langle C_G(\rho_1), C_G(\rho_2)\rangle$ contains a subgroup isomorphic to $\mathrm{F}_4(2)$. As $P \cong \mathrm{F}_4(2)$, we obtain

Lemma 11.3. $\langle C_G(\rho_1), C_G(\rho_2)\rangle = P$. $\square$

Lemma 11.4. *M normalizes P.*

Proof. Since $P \cong \mathrm{F}_4(2)$ and ρ_1 and ρ_2 are not conjugate in P, we have that $M \cap P = RS\langle f\rangle$. If $M \leq P$, we have nothing to do. If $M > M \cap P = RS\langle f\rangle$, then, by Lemma 4.8, there is an involution t of $M \setminus M \cap P$ such that $\rho_1^t = \rho_2$. This element normalizes P by Lemma 11.3. Thus M normalizes P. $\square$

Define $P_0 = PM$.

Lemma 11.5. *P_0 is strongly 3-embedded in G.*

Proof. Since $P \cong \mathrm{F}_4(2)$, there are three conjugacy classes of elements of order 3 in P and they are all witnessed in J. For $\langle x\rangle \in \mathcal{S}(J)$, we have $N_G(\langle x\rangle) = M \leq P_0$ and for $\langle x\rangle \in \mathcal{M}(J) \cup \mathcal{P}(J)$ we have $C_G(x) = C_P(x)$ by Lemma 8.2. Since also $N_G(S) \leq M \leq P_0$ we have P_0 is strongly 3-embedded in G by [8, Lemma 17.11]. $\square$

We can now determine the structure of P_0.

Lemma 11.6. *We have P_0 contains a Sylow 2-subgroup of G and either $P_0 = P$ or $P_0 \cong \mathrm{Aut}(\mathrm{F}_4(2))$.*

Proof. Assume that $T \notin \mathrm{Syl}_2(G)$ and let $T_1 > T$ normalize T. Then T_1 normalizes $Z(T) = \langle r_1, s_2\rangle$. Since $K_1 \leq P$ and $U_2 \leq P$, there exists $x \in T_1$ such that $r_1^x \neq r_1$ and $s_2^x \neq s_2$. Since $Z(T)$ has order 4, we deduce that $r_1^x = s_2$ and thus that $K_1^x = U_2$. Hence x normalizes $P = \langle K_1, U_2\rangle$ and $P_0 = P\langle x\rangle \cong \mathrm{Aut}(\mathrm{F}_4(2))$.

Now let $T_0 \in \mathrm{Syl}_2(P_0)$ ($P_0 = P$ or $P_0 = \mathrm{Aut}(P)$) and assume that $w \in N_G(T_0)$. As $r_1 \in T' \le T_0' \le T$, we have $r_1^w \in T \le P$. Employing Lemma 2.21 we see that all involutions of P commute with elements of order 3. By Lemma 11.5 $C_{P_0}(r_1^w)$ contains a Sylow 3-subgroup of $C_G(r_1^w)$. Hence it follows that $r_1^w \in r_1^{P_0} \cup s_2^{P_0}$. Then there is $x \in P_0$ such that $r_1 = r_1^{wx}$ or $s_2 = r_1^{wx}$. Since $\langle K_1, U_2\rangle = P$, we have $wx \in P$. However this means $w \in P_0$ and we infer $T_0 \in \mathrm{Syl}_2(G)$. $\square$

Now we produce the normal subgroup G_* with $G_* \cap P_0 = P$.

Lemma 11.7. *If $P_0 > P$, then G has a subgroup G_* of index 2 with $P = P_0 \cap G_*$. Furthermore G_* satisfies the hypothesis of Theorem 1.2.*

Proof. We let $T_0 \in \mathrm{Syl}_2(P_0)$ and $T \in \mathrm{Syl}_2(P)$ with $T_0 > T$. Suppose that $t \in T_0$ is an involution and $C_{P_0}(t)$ has a non-trivial Sylow 3-subgroup D. Then as P_0 is strongly 3-embedded by Lemma 11.5 we have that $D \in \mathrm{Syl}_3(C_G(t))$. Now by Lemma 2.21 P has four conjugacy classes of involutions and their centralizers have 3-parts of their orders 3^4, 3^4, 3^2 and 3^2. On the other hand, if we let $x \in T_0 \setminus T$ with $C_{P_0}(x) \cong 2 \times {}^2\mathrm{F}_4(2)$, then $C_P(x)$ has Sylow 3-subgroups which are extraspecial of order 3^3. It follows that x is not conjugate to any element in T and consequently G has a subgroup G_* of index 2 by Thompson's Transfer Lemma [8, Lemma 15.16]. Obviously then $P_0 \cap G_* = P$ and G_* satisfies the hypothesis of Theorem 1.2. $\square$

We finally prove that $G \cong \mathrm{F}_4(2)$ or $\mathrm{Aut}(\mathrm{F}_4(2))$.

Proof of Theorem 1.2. By Lemma 11.7, we may suppose that $P = P_0$. Using Lemma 2.21, P has exactly four conjugacy classes of involutions and each such involution t has $|C_P(t)|_3 \neq 1$. Since P is strongly 3-embedded in G, $C_P(t)$ contains a Sylow 3-subgroup of $C_G(t)$. Thus, as $|C_P(r_1)|_3 = 3^4$, we have $r_1^G \cap P \subseteq r_1^P \cup r_2^P$. Since r_1 and r_2 are not G-conjugate by Lemma 7.3 and 11.7, we get that $r_1^G \cap P = r_1^P$. We note that if N is a non-trivial normal subgroup of G, then, as $C_G(r_1) \le P$ and $r_1 \notin Z(P)$, $1 \neq C_N(r_1) \le N \cap P$ which means that $P \le N$. Because $N_G(S) \le P$, the Frattini Argument implies $G = N_G(S)N \le PN = N$. Hence G is a simple group. Now an application of Lemma 2.20 and the observation that P is neither soluble nor an alternating group yields $G = P$ and the proof is complete. $\square$

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